Box Pleating is Hard

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In their seminal 1996 paper, Bern and Hayes initiated investigation into the computational complexity of origami [3]. They proved that it is NP-hard to determine whether a given general crease pattern can be folded flat, when the creases have or have not been assigned crease directions (mountain fold or valley fold). Since that time, there has been considerable work in analyzing the computational complexity of other origami related problems. For example, Arkin et al. [1] proved that deciding foldability is hard even for simple folds, while Demaine et al. [4] proved that optimal circle packing for origami design is also hard. At the end of their paper, Bern and Hayes pose some interesting open questions to further their work. While most of them have been investigated since, two in particular (problems 2 and 3) have remained untouched until now.

First, while the gadgets used in their hardness proof for unassigned crease patterns are relatively straightforward, their gadgets for assigned crease patterns are considerably more convoluted, and quite difficult to check. In particular, we found an error in their crossover gadget where signals are not guaranteed to transmit dependably for wires that do not cross orthogonally, which is required in their construction. Is there a simpler way to achieve a correct result (i.e. "without tabs")?

Second, their reductions construct creases at a variety of unconstrained angles. Is deciding flat foldability easy under more restrictive inputs? For example *box pleating*, folding only along creases aligned at multiples of 45° to each other, is a subset of particular interest in transformational robotics and self-assembly, with a universality result constructing arbitrary polycubes using box pleating [2].

In this paper we prove deciding flat foldability of box pleated crease patterns to be NP-hard in both the unassigned and assigned cases, using relatively simple gadgets containing no more than 20 layers at any point. A *crease pattern* is a straight-line graph of creases on a square paper which all must be folded by ±180° resulting in a *flat folding*, a piecewise isometry in the plane such that the paper does not intersect itself. We call a crease a *valley fold* if it folds by 180° in the flat folding and a *mountain fold* if it folds by −180°. In the figures, mountain folds are drawn in red while valley folds are drawn in blue. If a crease may be either a mountain or valley fold in a flat folding, we call it *unassigned*. Alternatively, if we must find a flat folding consistent with a given assignment of each crease to either mountain or valley, the creases are *assigned*. In this abstract, we present the gadgets used in our reductions, but do not detail the proofs given limited space.

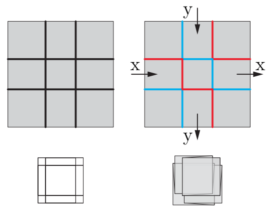

Figure 1: Unassigned crossover gadget and its only folded state up to symmetry.

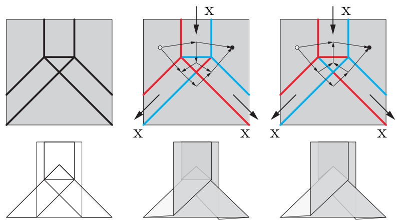

Figure 2: Unassigned split gadget and its only two valid folded states up to symmetry.

1 Unassigned Crease Patterns

Theorem 1 *Deciding flat foldability of box pleated crease patterns with unassigned creases is NP-complete.*

As in [3], the reduction is from Not-All-Equal 3-SAT (NAE 3-SAT): given a collection of boolean clauses, each containing exactly three literals, determine whether there exists a truth assignment such that each clause has either one or two true literals. However, our gadgets use only creases lying at multiples of 45° from each other and thus exist in the box pleating paradigm. A signal wire consists of two parallel creases (a *pleat*) one mountain and one valley. The assignment of which is which determines whether the signal is true or false. A crossover gadget, splitter gadget, and Not-All-Equal clause gadget are shown in Figures 1, 2, and 3 respectively. Figure 4 depicts an example of the reduction applied to a simple NAE 3-SAT instance.

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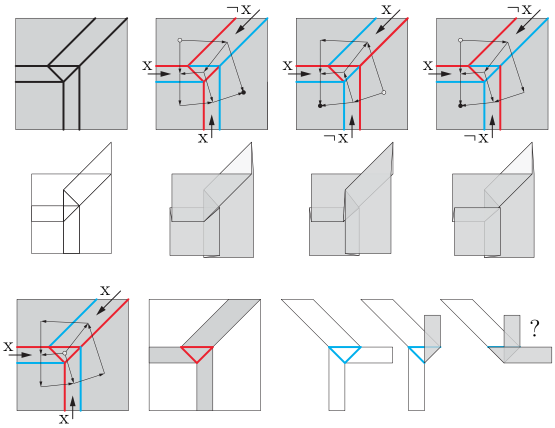

Figure 3: Unassigned clause gadget, including illustrations for why an All-Equal assignment of the variables is impossible.

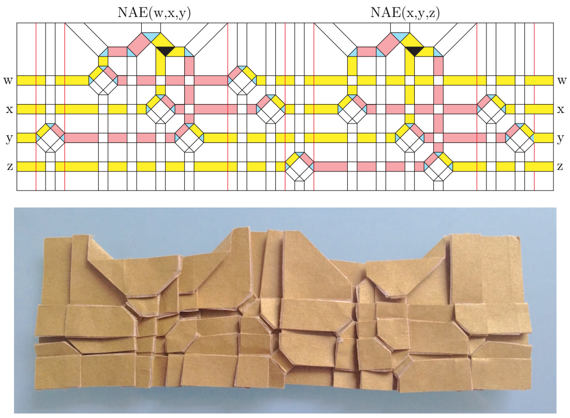

Figure 4: [Above] An example of the reduction applied to a simple NAE 3-SAT instance. [Below] A physical folding of this pattern demonstrating the satisfying solution (w,x,y,z) = (T,F,T,T).

2 Assigned Crease Patterns

Theorem 2 *Deciding flat foldability of box pleated crease patterns with assigned creases is NP-complete.*

Again, the reduction is from NAE 3-SAT. Here, a signal wire consists of four parallel creases with given assignment either mountain or valley. The layer ordering of which pleat is on top determines whether the signal is true or false. A crossover gadget, splitter gadget, and Not-All-Equal clause gadget are shown in Figures 5, 6, and 7 respectively. Figure 8 depicts an example of the reduction applied to a simple NAE 3-SAT instance.

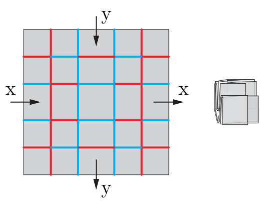

Figure 5: Assigned crossover gadget and its only folded state up to symmetry.

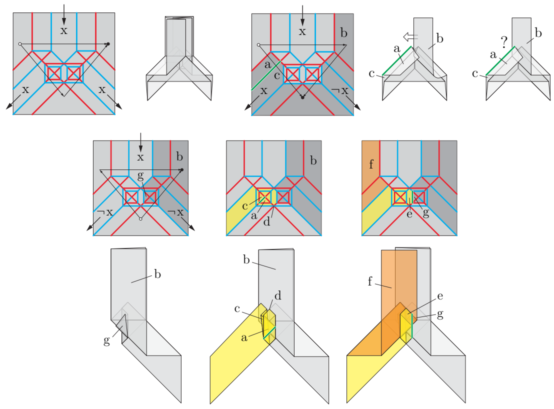

Figure 6: Assigned split gadget, including illustrations for why non-splitting assignments of the variables is impossible.

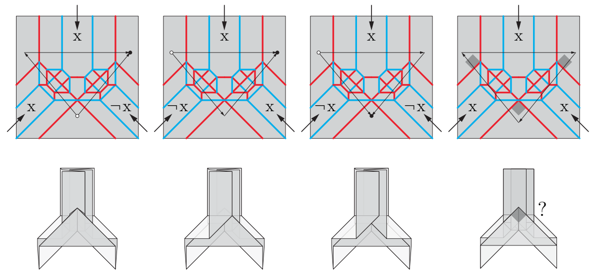

Figure 7: Assigned clause gadget and its only three valid folded states up to symmetry, including an illustration for why an All-Equal assignment of the variables is impossible.

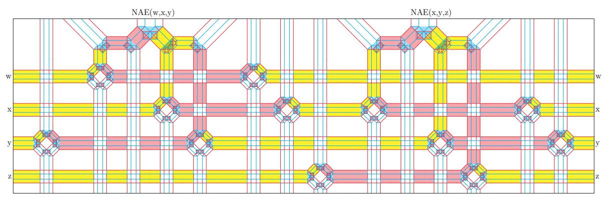

Figure 8: An example of the reduction applied to a simple NAE 3-SAT instance.

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