Efficient Foldings of Convex Polyhedra from Convex Paper

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Can one fold a given polyhedral surface from a convex polygon? A universality result first presented at the 3rd CGC Workshop on Computational Geometry in 1998 [DDM00] proved the answer is always yes, though the construction is highly inefficient with respect to the diameter of the polygon required. While their strip method approaches the optimal bound on area, it requires a polygon whose diameter grows with feature size. Other constructions such as Origamizer appear to be more efficient in practice, though have not been proven to always converge to an efficient solution [Tac10].

This question arises naturally in the fields of engineering and computational geometry, with applications to packaging and sheet metal bending. Current packaging research investigates optimal layout for die-cut unfoldings [LT07] as well as path planning for how to fold complex wrappings without self-intersection [YD08]. Fabrication processes that utilize sheet metal bending also need to solve this problem [WC95].

We consider a special case of this problem by presenting a method for folding convex polyhedra from convex polygons. Our method builds on existing work about exact[1] foldings of convex polyhedra from nonconvex polygons, traditionally studied from the point of view of unfoldings [DO07]. Our construction starts from a source unfolding [SS86, Ale05] of the target polyhedron and constructs a folding of the polyhedron from the source unfolding's convex hull. Our main result is the following:

Theorem 1 *The convex hull of any source unfolding of a convex polyhedron can be folded to cover the polyhedron and nothing else, without self intersection.*

At a high level, we prove this theorem by constructing such a folded state from the convex hull of any source unfolding. The approach is to map the source unfolding S contained in the convex hull H onto the polyhedron P while collapsing the extra paper in $H \setminus S$ onto the surface of P. The source unfolding (f, Λ) consists of a folding map $f : H \to P$ and a layer order $\Lambda : H \times H \to \pm 1$. The bulk of the construction is showing how to fold the paper in $H \setminus S$ flat against P. The construction relies heavily on properties of the source unfolding [SS86, Ale05]. The extra paper $H \setminus S$ is a (possibly empty) set of disjoint regions. Each region is contained in its own acute sector $V \subset H$ incident to the source. Such a region T is a polygonal tree satisfying certain special properties. We exploit these properties to collapse it onto the subset of $f(V \cap S)$ adjacent to one side of the cut tree where T connects to S. Our folding maps points in V to points in $f(V \cap S)$, so layer ordering Λ can be assigned locally, and self intersection can be avoided. Examples of the construction applied to a regular dodecahedron are shown in Figure 1.

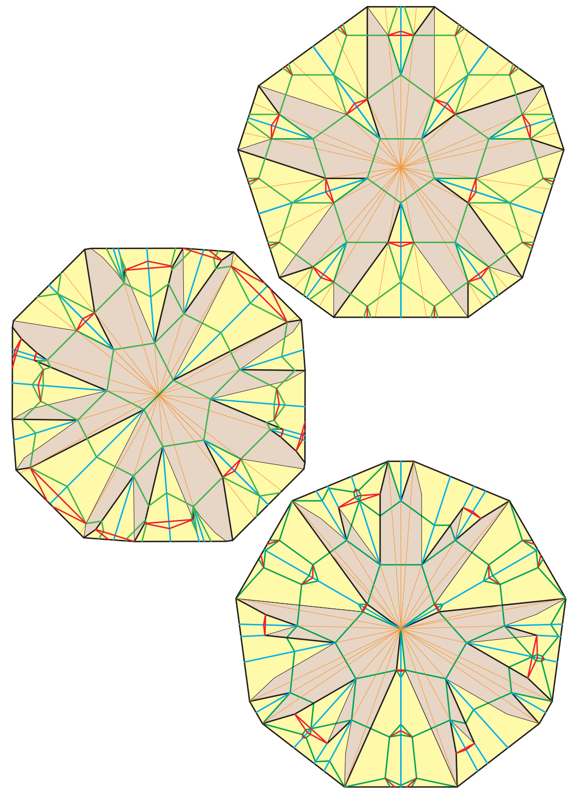

Figure 1: Examples of foldings of a regular dodecahedron produced by the algorithm. The generating source unfolding is shown in gray. Thick lines are folded completely (±180°) except for green lines which fold to the fold angle ≈ 63.4° of the regular dodecahedron. The top unfolding is sourced from the center of a face, middle unfolding from the center of an edge, and bottom unfolding from a vertex.

The constructed folding is efficient in that the diameter of the folding is within a constant factor of the diameter of a folding with minimum diameter, while the ratio of the constructed folding's area to the area of a folding with minimum area depends only linearly on the feature ratio of the polyhedron.

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[1] By contrast, our notion of folding is not an exact covering, allowing multiple layers at a point on the target polyhedron.

Theorem 2 *Given a convex polyhedron P, let H be the convex hull of a source unfolding of P, and let OPT be a polygon with minimum diameter that can be folded to cover the surface of P and nothing else. Then* $\operatorname{diam}(H)/\operatorname{diam}(OPT) = O(1)$.

Proof. Let R be the circumradius of P. The diameter of OPT must be at least R or else any perpendicular projection onto P of any radius from the circumcenter to a vertex could not be covered by OPT. Further, the shortest path along the surface of P between any two points on the surface is at most πR, since P is convex, and πR is the longest distance between points on the smallest sphere containing P. Because the source unfolding is constructed from shortest paths from the source, it must be contained in a circle of radius πR. Thus, $\operatorname{diam}(H)/\operatorname{diam}(OPT) \leq 2\pi = O(1)$. $\square$

Theorem 3 *Given a convex polyhedron P, let ρ be the minimum feature ratio R/r between the circumradius R and the inradius r of P. Let H be the convex hull of a source unfolding of P, and let OPT be a polygon with minimum area that can be folded to cover the surface of P and nothing else. Then* $\operatorname{area}(H)/\operatorname{area}(OPT) = O(\rho)$.

Proof. First, the center p of the insphere of P is at least distance R from some vertex v of P, or else a sphere with radius smaller than R could enclose all the vertices of P. By convexity, there is an isosceles triangle in P, with altitude from p to v and a base of length $2r$ centered at p. The area of this triangle is then greater than Rr. The orthogonal projection of this triangle onto P's surface marks a subset of P that is at least as large, so the surface area of P, and thus the area of OPT, are at least Rr. Second, by the same argument as in the proof of Theorem 3, the source unfolding has radius at most πR, so the area of H is at most $\pi^3 R^2$. Thus, $\operatorname{area}(H)/\operatorname{area}(OPT) \leq \pi^3 R/r = O(\rho)$. $\square$

This bound is tight in that there are source unfoldings which attain this bound; see Figure 2. However, we imagine a smart choice of source may lead to a source unfolding convex hull whose area is closer to optimal.

Additionally, the number of creases needed for our construction has a pseudopolynomial bound.

Theorem 4 *Let α be the smallest angle that can be formed from two adjacent vertices of the source unfolding used in the construction and its source. Then the number of creases in f is bounded by $O(n^2/\alpha)$.*

This bound depends on a measure of the source unfolding chosen for the construction and not on the input polyhedron itself. We conjecture there is a more direct connection between this minimum angle and the polyhedron such as maximum or minimum solid angle at a vertex.

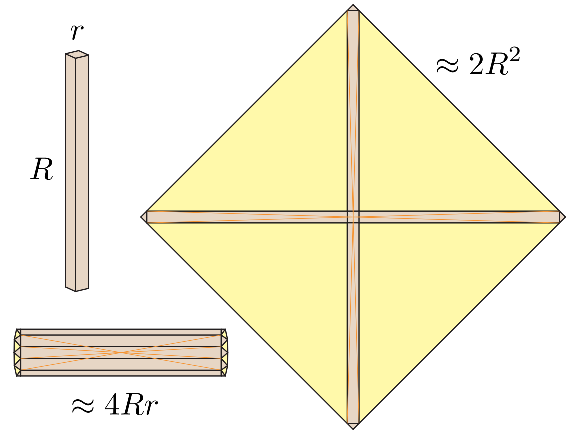

Figure 2: A polyhedron with two source unfoldings whose convex hulls have areas that differ by factor proportional to $\rho = R/r$ for $R \gg r$. The source for the left source unfolding is the center of a long face while the source for the right source unfolding is the center of a small face. Source unfolding is shown in gray, with convex hull colored in yellow.

References

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