Satisfying Multiple Boundary Conditions

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It has been shown that an isometry always exists to fold a paper to match a non-expansive folding of its boundary [1]. However, there is little (if any) research in designing crease patterns that satisfy multiple constraints. In this paper, we analyze crease patterns that can fold to multiple prescribed folded boundaries and flat-foldable states, such that every crease in the crease pattern is finitely folded in each folding.

Theorem 1 *Given a four cornered paper, there exists a single vertex crease pattern folding through each corner of the paper that also folds flat.*

Proof. A single degree-four vertex in a flat-foldable crease pattern must obey Kawasaki's theorem, that the sum of opposite angles sum to π. From this condition, one can derive a condition on possible positions (x, y) of the single vertex. We can parameterize any simple quadrilateral with cyclically ordered points $a = (-1, 0)$, $b = (x_1, y_1)$, $c = (1, 0)$, and $d = (x_2, y_2)$, where y_1 is positive and y_2 is negative, and the line from a to c is a visible diagonal. The condition on the location of a flat-foldable vertex is then given by the following cubic equation:

$$x(y_1 + y_2)(x^2 + y^2 - 1) - y(x_1 + x_2)(x^2 + y^2 + 1) + (x_1 y_2 + x_2 y_1)(y^2 - x^2 + 1) + 2xy(1 + x_1 x_2 - y_1 y_2) = 0. \tag{1}$$

The curve defined by this equation passes through each corner of the paper, as can be readily verified. However, we must prove that the curve passes through the interior of the paper. It suffices to show that the tangent to the curve at one of the vertices passes between its two adjacent edges. Taking partial derivatives of Equation 1, one can show the tangent to the curve at p_a has the same direction the following vector:

$$v_T = (\,(x_1 + 1)(x_2 + 1) - y_1 y_2,\ y_1(x_2 + 1) + y_2(x_1 + 1)\,). \tag{2}$$

The edges adjacent to p_a have directions $v_b = (x_1 + 1, y_2)$ and $v_d = (x_2 + 1, y_2)$ respectively. Taking magnitude of the cross products in the $\hat{z}$ direction out of the plane yields the following relations:

$$(v_T \times v_b) \cdot \hat{z} = -(\,(1 + x_1)^2 + y_1^2\,)y_2; \tag{3}$$

$$(v_T \times v_d) \cdot \hat{z} = -(\,(1 + x_2)^2 + y_2^2\,)y_1. \tag{4}$$

Because y_2 is always negative, the first condition is always positive, so the top edge is a left turn from the tangent line. Because y_1 is always positive, the second condition is always negative, so the bottom edge is a right turn from the tangent line, so local to p_a, the curve must intersect the quadrilateral, completing the proof. □

For quadrilateral paper, the solution space of single vertex crease patterns satisfying a folding of its boundary is an ellipse on the interior of the paper. The equation of this ellipse in general is quite complicated. However, in the case of kite quadrilaterals, the ellipse is axis aligned with the diagonals, and for squares the ellipse is centered. Let the corners of the square be $(-1, 0)$, $(0, 1)$, $(1, 0)$ and $(0, -1)$. We parameterize the folding of the square boundary by the distances between the two diagonals, distance $2\sqrt{1 - a^2}$ along the x axis and distance $2\sqrt{1 - b^2}$ along the y axis; this parameterization will simplify the equations later on. Using this parameterization, the equation of the ellipse of crease pattern vertices satisfying the boundary condition (a, b) is as follows:

$$\frac{x^2}{a^2(1 - b^2)} + \frac{y^2}{b^2(1 - a^2)} = a^2 + b^2. \tag{5}$$

Since the ellipse is centered at the origin, it must cross the x and y axes four times except in the degenerate cases where the ellipse becomes a line or a point, specifically when a or b equal 1 or 0.

Theorem 2 *Given a square of paper and two foldings (a_1, b_1), (a_2, b_2) of its boundary folded only at the corners, then if the intervals $[a_1, b_1]$ and $[a_2, b_2]$ overlap, then there exists a single vertex crease pattern that folds exactly to both boundaries.*

Proof. The proof is by construction. The approach will be to calculate the set of possible crease patterns with one interior vertex that folds to each boundary, and show that the two sets have crease patterns in common when the intervals $[a_1, b_1]$ and $[a_2, b_2]$ overlap.

We have already shown that the solution space for each boundary condition is an ellipse given by Equation 5. Given two such ellipses parameterized by (a_1, b_1) and (a_2, b_2), they can be made to intersect as long as the smaller major radius is larger

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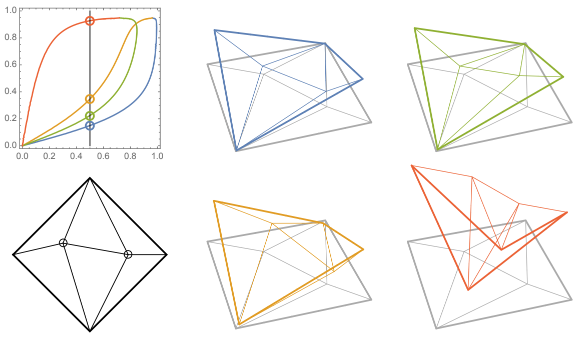

Figure 1: The state space of a two vertex crease pattern, plotting a vs. b, and four folded states for a single value of a.

than the minor radius of the other since the roles of a and b are interchangeable under boundary mappings. A simple yet tedious case analysis shows that this equation holds when intervals $[a_1, b_1]$ and $[a_2, b_2]$ overlap. The converse statement is not true as there are points when the intervals do not overlap such that the inequality is still true. □

Theorem 3 *Given any two nonexpansive boundary foldings of a square paper folding at its vertices, there exists a one or two-vertex crease pattern that can fold rigidly to meet both boundary conditions.*

Proof. The proof is by construction. The approach will be to calculate a subset of possible crease patterns with two interior vertices that folds to each boundary, and show that the two sets have crease patterns in common.

We will parameterize a subset of two-vertex crease patterns in the special case where one vertex resides on a diagonal. We will let s be the distance between this vertex p and point $(-1, 0)$. Solving the distance equations again yields the equation of an ellipse, this time of the following form:

$$\frac{(x - x_0)^2}{r_x^2} + \frac{y^2}{r_y^2} - 1 = 0. \tag{6}$$

However, this time there are two possible ellipses for each choice of boundary condition: one when the crease from $(-1, 0)$ to p is a valley fold, and one when the crease is a mountain fold. The parameters of the ellipse in each case are given by:

$$x_0 = \frac{sb^2(1 + b^2)}{2(a^2 + b^2)((1 - s) + b^2) \pm 2sb\sqrt{1 - a^2}\sqrt{a^2 + b^2}}; \tag{7}$$

$$r_x = b\sqrt{\frac{1 - a^2}{a^2 + b^2}} - \frac{sb^2(1 - b^2)}{2(a^2 + b^2)((1 - s) + b^2) \pm 2sb\sqrt{1 - a^2}\sqrt{a^2 + b^2}}; \tag{8}$$

$$r_y = a\sqrt{\frac{1 - b^2}{a^2 + b^2}}\left(1 - \frac{sb^2}{(a^2(1 - s) + b^2) \pm b\sqrt{1 - a^2}\sqrt{a^2 + b^2}}\right). \tag{9}$$

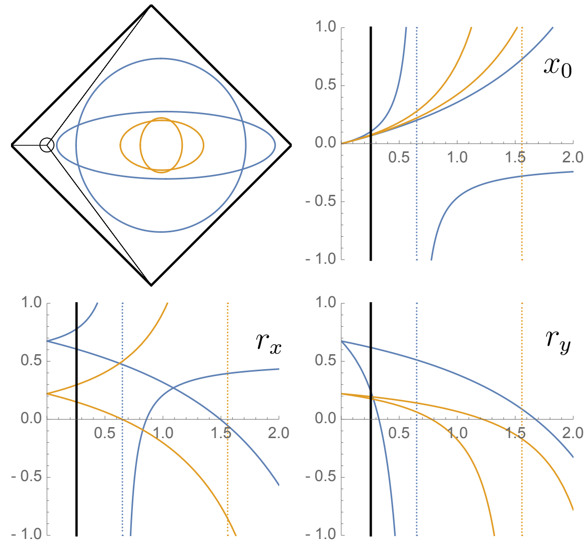

Figure 2: Graphs of how x_0, r_x, and r_y vary with respect to s for $(a_1, b_1) = (0.3, 0.3)$ (blue) and $(a_2, b_2) = (0.95, 0.95)$ (orange).

When $s = 0$, these parameters reduce Equation 6 to Equation 5. Figure 1 shows a plot of this state space for one such crease pattern. The bottom right corner corresponds to the flat state.

The equations continue to define an ellipse as long as r_y does not become negative. If the ellipse corresponding to (a_1, b_1) and (a_2, b_2) do not intersect at $s = 0$, then that means both r_x and r_y are larger for one and not the other because x_0 is zero. Without loss of generality, assume $a_1 > a_2$ and $b_1 > b_2$. r_y is zero precisely when:

$$s(r_y = 0) = 1 \pm b\sqrt{\frac{1 - a^2}{a^2 + b^2}}. \tag{10}$$

Since r_y is symmetric about 1 for any (a, b), this means r_y for (a_1, b_1) and r_y for (a_2, b_2) must be equal form some s. If they are equal, their corresponding ellipses must intersect, which corresponds to a two vertex solution. Figure 2 shows how x_0, r_x, and r_y vary with respect to s. □

We conjecture that any finite set of corner foldings of a square can be satisfied with a finite crease pattern such that every crease folds by a nonzero amount when satisfying each boundary folding.

References

[1] E. D. Demaine and J. Ku. Filling a hole in a crease pattern: Isometric mapping of a polygon given a folding of its boundary. In *Origami[6]: Proceedings of the 6th International Meeting on Origami in Science, Mathematics and Education (OSME 2014)*, page to appear. Tokyo, Japan, August 10–13 2014.