Infinite All-Layers Simple Foldability

Hugo Akitaya* Cordelia Avery† Joseph Bergeron†
Erik D. Demaine† Justin Kopinsky† Jason Ku†

A classic problem in computational origami is *flat foldability*: given a crease pattern (planar straight-line graph with n edges) on a polygonal piece of paper P, can P be folded flat isometrically without self-intersection while creasing at all creases (edges) in the crease pattern? The problem can also be defined for *assigned* crease patterns, in which every crease is labeled *mountain* or *valley* depending on the direction it is allowed to fold. The decision problem (for both assigned or unassigned) is NP-hard [5], even when the paper is an axis-aligned rectangle and the creases are at multiples of 45° [2]. But even when a crease pattern does fold flat, the motion to achieve that folding can be complicated [6], making the process impractical in some physical settings.

Motivated by practical folding processes in manufacturing such as sheet-metal bending, Arkin et al. [3] introduced the idea of *simple foldability*—flat foldability by a sequence of simple folds. Informally, a *simple fold* is defined by a line segment and rotates a portion of the paper around this segment by ±180°, while avoiding self-intersection. The problem generalizes to *d*-dimensions. In particular, for 1D paper, P is a line segment and creases are defined by points in P. In [3], they defined several models for simple folds and, for many models, showed that deciding simple foldability is polynomial for 1D paper, polynomial for rectangular paper with axis-aligned creases, weakly NP-complete for rectangular paper with creases at multiples of 45°, and weakly NP-complete for orthogonal paper with axis-aligned creases. In particular, they provided an algorithm to determine simple foldability of a 1D paper in $O(n \log n)$ deterministic time and $O(n)$ randomized time in the *all-layers* model, requiring that a simple fold through one crease, also folds through all layers overlapping that crease. Akitaya et al. [1] extended the list of simple folding models, and for many models showed simple foldability to be strongly NP-hard for 2D paper. In particular, they introduced the *infinite all-layers* model of simple folds for 2D paper which is studied here, requiring that each simple fold be defined by an infinite line, and that all layers of paper intersecting this line must be folded. This model is probably the most practical simple folding model; for example, Balkcom's robotic folding system [4] is restricted to this model.

In this paper, we improve on [3] giving a deterministic $O(n)$-time algorithm to decide simple foldability of 1D crease patterns in the all-layers model. Then, we prove two results concerning the complexity of one of the few remaining open problems in this area [1]: infinite all-layers simple foldability on *orthogonal crease patterns*, axis-aligned orthogonal 2D paper with axis-aligned creases. First, we prove that this problem can be solved in linear time when creases are *fully unassigned*. On the other hand, when the creases are *partially assigned* (some creases must fold mountain, some creases must fold valley, while others can freely fold mountain or valley), this problem becomes strongly NP-complete, even for an axis-aligned rectangle of paper.

Theorem 1 *All-layers simple foldability of a 1D crease pattern can be decided in deterministic linear time.*

Proof sketch: We reduce the problem to a "string folding" problem as in [3], representing the input as a string of the form $\ell_0 d_1 \ell_1 d_2 \dots d_{n-1} \ell_{n-1} d_n \ell_n$ where each $d_i \in \{M, V\}$ represents the assignment of the i-th crease and $\ell_i \in \mathbb{R}$ represents the length of the i-th uncreased line segment in P. For an instance to be simple foldable in this model, any fold must map a crease onto another crease of opposite assignment. After a fold is performed, we obtain a smaller crease pattern by ignoring paper overlap. By [3] the smaller crease pattern is simple foldable if and only if the original one is. The *size* of a fold is defined by the difference on the length of the strings representing the crease patterns. We adapt the algorithm in [7] to recognize the smallest possible fold in a crease pattern that runs in linear time on the size of the output fold, leading to an amortized linear time algorithm overall. Unassigned crease patterns can also be solved by a simple modification of this algorithm.

Theorem 2 *Infinite all-layers simple foldability of a fully unassigned orthogonal crease pattern can be decided in deterministic linear time.*

Proof sketch: We first provide a linear time reduction of infinite all-layers simple foldability of unassigned orthogonal crease patterns to instances on rectangular paper. Such instances are equivalent in the finite and infinite all-layers models [2]. We then reduce the problem on a rectangle to two instances of 1D simple foldability which by Theorem 1 can each be decided in deterministic linear time.

*Tufts University, hugo.alves_akitaya@tufts.edu
†CSAIL, Massachusetts Institute of Technology, {cavery, jbergero,edemaine,jkopin,jasonku}@mit.edu

Theorem 3 *Deciding infinite all-layers simple foldibility of partially assigned orthogonal crease patterns is NP-Complete, even for creases on a square grid on rectangular paper.*

Proof sketch: The problem is in NP as a valid sequence of simple folds represents a certificate of at most linear size that can be checked in polynomial time. We show NP-hardness via a reduction from 3SAT. Given an instance of 3SAT on n variables and m clauses, we build a partially assigned simple foldability instance as illustrated in Figure 1. Informally, yellow dots on the same vertical line represent a clause. The partial assignment forces any legal sequence of simple folds to fold through either t_1 or f_1 but not both, forcing a yellow dot onto either a green or red dot respectively, encoding the boolean assignment of the variable x_1. A vertical fold on the right edge of the paper must occur next, followed by folding along either t_1' or f_1', consistent with whether t_1 or f_1 was initially chosen. These folds force the yellow dots directly below t_1 and f_1 to coincide with the yellow dots directly above t_1 and f_1, and adds a valley assignment to a crease incident to a yellow dot if its corresponding clause contains a literal involving x_1 that evaluates to FALSE. We apply induction on the resulting crease pattern to bring all the yellow dots to lie on top of the m upper-most yellow dots. A crease pattern containing a vertex incident to only valley creases is not flat foldable. After folding as described for any given assignment of the variables, we show that the each resulting yellow dot will be incident to at least one non-valley crease if and only if the SAT instance has a positive solution, and that the resulting crease pattern can be folded by a sequence of infinite all-layers simple folds.

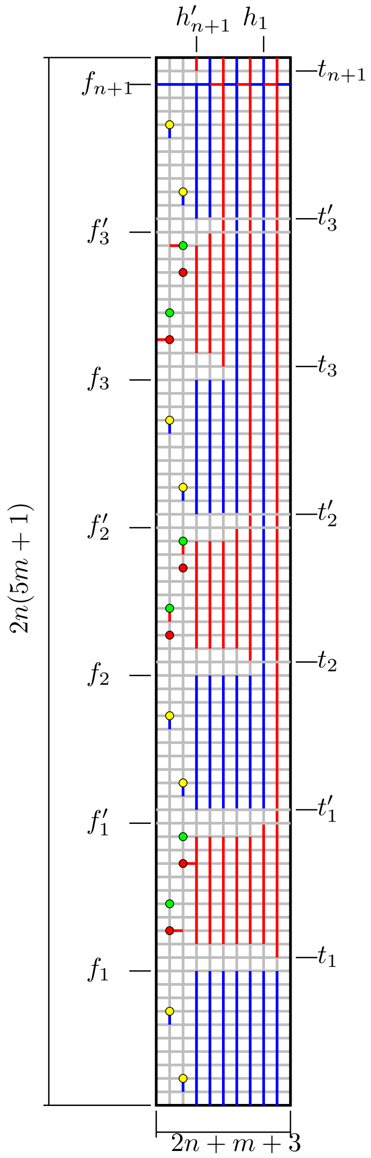

Figure 1: Reduction from the instance $(x_1 \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_3})$ of 3SAT to partially assigned simple foldability under the infinite all-layers model.

Acknowledgments. This work was initiated during an open problem session associated with MIT class 6.849 (Geometric Folding Algorithms) taught by E. Demaine. We thank the other participants of that class for providing a stimulating research environment. Supported in part by NSF grants EFRI-1240383, CCF-1138967 and CCF-1422311, and Science without Borders.

References

[1] H. Akitaya, E. D. Demaine, and J. S. Ku. Simple folding is really hard. *Journal of Information Processing*. To appear. Special issue of papers from the 19th Japan Conference on Discrete and Computational Geometry, Graphs, and Games, September 2016.

[2] H. A. Akitaya, K. C. Cheung, E. D. Demaine, T. Horiyama, T. C. Hull, J. S. Ku, T. Tachi, and R. Uehara. Box pleating is hard. In *Revised Papers from the 18th Japan Conference on Discrete and Computational Geometry and Graphs*, pages 167–179. 2015.

[3] E. M. Arkin, M. A. Bender, E. D. Demaine, M. L. Demaine, J. S. B. Mitchell, S. Sethia, and S. S. Skiena. When can you fold a map? *Computational Geometry: Theory and Applications*, 29(1):23–46, 2004.

[4] D. J. Balkcom and M. T. Mason. Robotic origami folding. *The International Journal of Robotics Research*, 27(5):613–627, 2008.

[5] M. Bern and B. Hayes. The complexity of flat origami. In *Proceedings of the 7th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 175–183, 1996.

[6] E. D. Demaine and J. S. B. Mitchell. Reaching folded states of a rectangular piece of paper. In *Proceedings of the 13th Canadian Conference on Computational Geometry*, pages 73–75, Waterloo, August 2001.

[7] G. Manacher. A new linear-time "on-line" algorithm for finding the smallest initial palindrome of a string. *J. ACM*, 22(3):346–351, July 1975.