Lecture Topics (2/5, 2/7, 2/12): Quantum operations; quantum error correction criteria; CSS codes

Recommended Reading: Nielsen and Chuang, Sections 4.2-4.4, 8.1-8.3, 10.1-10.4

Problems:

P1: (Operator Sum Representation: Examples) In class, we learned that the interaction of any quantum system with an environment can be mathematically expressed by a quantum operation, \( \mathcal{E}(\rho) \), defined as
\[
\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger,
\]
where the only condition on the operation elements \( E_k \) is that \( \sum_k E_k E_k^\dagger = I \). This is known as the operator sum representation (OSR). Here, we explore some of the physics implied by this model, and study some important examples introduced in the lecture.

(a) You are given a black box which takes single qubit states \( \rho_{in} \) as input, and produces \( \rho_{out} \) as output. Suppose the box just replaces its input with \( |0\rangle \), such that \( \rho_{out} = |0\rangle \langle 0| \). Give a set of operation elements \( \{E_k\} \) describing the black box, such that \( \rho_{out} = \sum_k E_k \rho_{in} E_k^\dagger \).

(b) Suppose the black box replaces any input \( \rho_{in} \) with the completely randomized state \( I/2 \). Give \( \{E_k\} \) describing the operation of this box. It is amazing that quantum codes can correct for this single qubit erasure error, which completely destroys the original qubit!

(c) From performing experiments, you determine that the black box performs \( \mathcal{E}(\rho_{in}) = pI/2 + (1 - p)\rho_{in} \), where \( p \) is a probability. Give \( \{E_k\} \) describing the operation of this box.

(d) You are given a schematic diagram for the quantum circuit inside the black box, which is

Each line represents a qubit; the top qubit is the qubit transferred into and out of the box, and
the bottom qubit is the “environment” to which the box is connected. Give \{E_k\} describing the operation of this box; write this in terms of \(\lambda\), where \(\cos(\theta/2) = e^{-\lambda}\).

(e) You are given another schematic diagram for the quantum circuit inside the black box, which is

\[
\begin{array}{c}
\rho_{\text{in}} \quad \text{Z} \quad \rho_{\text{out}} \\
\rho_{\text{env}}
\end{array}
\]

The environment state \(\rho_{\text{env}} = \alpha|0\rangle\langle0| + (1 - \alpha)|1\rangle\langle1|\), where \(\alpha\) is a probability. Give \{E_k\} describing the operation of this box. How is this box related to that in part (d)? If they are related, give an expression relating \(\alpha\) and \(\lambda\).

P2: (A 5-qubit quantum code) We saw in class that a quantum code defined by projector \(P = \sum_\ell |\psi_\ell\rangle\langle\psi_\ell|\), for codeword states \(|\psi_\ell\rangle\), corrects errors \(E_k\) if and only if \(PE_k^\dagger E_j P = d_k \delta_{jk}\), for some constant \(d_k\). Let the notation \(U_1 U_2 U_3 U_4 U_5\) denote the tensor product \(U_1 \otimes U_2 \otimes \cdots \otimes U_5\), as was used in class, where the \(U_j\) are Pauli matrices. Consider the matrix

\[
P = \frac{1}{16} \begin{bmatrix} 3 & & & & \\
& IZYYZ + ZYYZI + YYZIZ + YZIZY + ZIZYY & & & \\
& & IXZZX + XZZXI + ZZXIX + ZXIXZ + XIXZZ & & \\
& & & IYXXY + YXXYI + XXYIY + XYIYX + YIYXX & \\
& & & & + 2 (ZXYYX + XYYXZ + YYXZX + XZXYY + XZXYY) \rangle \\
& & & & -2ZZZZZ \end{bmatrix}
\]

Note that while this looks complicated, many of the terms are actually cyclic shifts of others.

(a) Show that \(P\) is a projection operator onto a 6-dimensional subspace of the 5-qubit Hilbert space.

(b) Show that the space defined by \(P\) defines a quantum code which can correct for any single qubit erasure error.

(c) Explain why the distance of the code is \(d = 2\), so that this is a \(((5,6,2))\) quantum code.

P3: (The CSS codes construction) Classical linear codes can be translated directly into quantum codes. In this exercise, we explore an example using the CSS construction.

(a) A binary linear code \(C\) encoding \(k\) bits of information into an \(n\) bit code space is a \(k\)-dimensional subspace of the vector space of \(n\)-bit strings. The \(2^k\) codewords in the code can be specified as the span of the columns of an \(n\) by \(k\) generator matrix \(G\) whose entries are zeros and ones. The codewords are given by \(Gx\) where \(x\) is a \(k\)-bit column vector representing the information string
to be encoded. The arithmetic operations are modulo 2. Let the generator matrix

$$G = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(3)

declare the code $C$. Write out $C$ explicitly. Use the Gauss-Jordan procedure to column reduce $G$ and arrive at the standard (or systematic) form

$$G = \begin{bmatrix} I \\ A \end{bmatrix}$$

(4)

where $A$ is an $n-k$ by $k$ matrix.

(b) Errors are detected by computing a collection of parity checks. The parity checks can be gathered into an $n-k$ by $n$ check matrix $H$ satisfying $Hx = 0$ for all codewords $x$. If $Hx' \neq 0$ for a vector $x'$, then $x'$ is not a codeword and an error has occurred. For a binary linear code in standard form $H = [IA]$. Verify that $Hx = 0$ for all the codewords of $C$. Prove that $HG = 0$ for a binary linear code.

(c) The minimum (Hamming) distance over all pairs of codewords of $C$,

$$d(C) = \min_{x,y \in C, x \neq y} d(x,y),$$

(5)

where $d(x,y)$ is the number of bits where $x$ and $y$ differ, is the minimum number of bit flips that can lead to an undetectable error. A code can detect $d-1$ single qubit errors and correct $\lfloor (d-1)/2 \rfloor$ single qubit errors. What is $d(C)$ for this example? How many errors does this code correct?

(d) The dual code $C^\perp$ of $C$ is the code with generator matrix $G^\perp = H^T$. Confirm that $C^\perp \subseteq C$ for this example. As a special case of the CSS construction, consider the quantum code that is the span of the states

$$|\psi(x)\rangle = \frac{1}{\sqrt{|C^\perp|}} \sum_{y \in C^\perp} |x+y\rangle$$

(6)

for $x \in C$. Show that these states must be orthogonal if $x \neq y \mod C^\perp$. Write the set of unique states $|\psi(x)\rangle$ for this example. Confirm that single qubit Pauli errors map these states to orthogonal states. If $k$ is the dimension of $C$ and $k^\perp$ is the dimension of $C^\perp$, what is $k - k^\perp$ for this example? How many qubits does this code encode?

(e) Let $C_{jk}$ be the CNOT gate applied on qubits $j$ and $k$, with $j$ being the control qubit. Show that $C_{15}C_{26}C_{37}C_{48}|\psi(1100)\rangle \otimes |\psi(1010)\rangle = |\psi(1100)\rangle \otimes |\psi(0110)\rangle$. Explain the action of this gate on $|\psi(x)\rangle \otimes |\psi(y)\rangle$. 

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