Motives with small conductor

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The four sections of today's talk

1. Small conductor problems

2. The explicit formula and lower bounds on conductors

3. The smallest conductors coming from special hypergeometric motives in low degree

4. A motive $M = M_6 \oplus M_8$ of particularly small conductor

1. Small conductor problems

Here are some important sequences:

1. Conductors of isogeny classes of elliptic curves are 11, 14, 15, 17, 19, 20, 21, 24, 26, 26, ...

2. Levels of weight two newforms on Γ_0 are 11, 14, 15, 17, 19, 20, 21, 23, 23, 24, 26, 26, ...

3. Levels of weight four newforms on Γ_0 are 5 ,6, 7, 8, 9, 10, 11, 11, 12, ...

4. Conductors of L-functions looking like they come from abelian surfaces with Sato-Tate group Sp_4 are 249, 277, 295, 349, 353, 388, 389, 394, ... (Farmer-Koutsoliotas)

5. Paramodular levels of rational degree two Siegel eigenforms with Sato-Tate group Sp_4 in weight three: 61, 73, 79, ... (Ash-Gunnells-McConnell; Poor-Yuen)

A conjectural setting

We would like to discuss similar sequences in a very broad context. To do this in a clean way, we we will assume the fundamental and widely believed conjecture relating motives, L-functions, and automorphic forms, centering on $L(s, M) = L(s, \pi)$.

More precisely, working in the analytic normalization, we assume that

irreducible degree d motives $M \in M(\mathbb{Q}, \mathbb{C})$ modulo Tate twisting,

are in bijection with

primitive Selberg class L-functions with real spectral parameters,

and these come bijectively from

cuspidal automorphic representations π of $GL_d(\mathbb{A})$ with algebraic infinity type.

Hodge vectors and Gamma factors

Since we are working in the analytic normalization, it is best to rewrite $h^{p,q}$ as h^{p-q}. We present Hodge numbers of a pure-parity motive M as a *Hodge vector*:

$$(h^{-w}, h^{2-w}, \ldots, h^{w-2}, h^{w})$$

with $h^k = h^{-k}$. When the parity is even, we consider Hodge vectors as coming with a refinement $h^0 = h^{0+} + h^{0-}$.

The Gamma factor of $L(M,s)$ is then

$$\Gamma_{\mathbb{R}}(s)^{h^{0+}} \Gamma_{\mathbb{R}}(s+1)^{h^{0-}} \prod_{k \geq 1} \Gamma_{\mathbb{C}}(s + \frac{k}{2})^{h^k}.$$

For simplicity, all our explicit examples will have odd parity and so there will be no $\Gamma_{\mathbb{R}}$ factors.

Sequences belonging to a fixed Hodge vector

As a catch-all, we have the sequence $s(h)$ of conductors of motives $M \in M(\mathbb{Q}, \mathbb{C})$ with Hodge vector h. We can consider variants, like demanding that coefficients be within say $\mathbb{Q}$ or $\mathbb{R}$, or demanding irreducibility, or demanding genericity of the Sato-Tate group. The five examples again:

1. $s_{\mathbb{Q}}(1,1)$: 11, 14, 15, 17, 19, 20, 21, 24, 26, 26, ...
2. $s_{\mathbb{R}}(1,1)$: 11, 14, 15, 17, 19, 20, 21, 23, 23, 24, 26, 26, ...
3. $s_{\mathbb{R}}(1,0,0,1)$: 5 ,6, 7, 8, 9, 10, 11, 11, 12, ...
4. $s_{\mathbb{Q}}^{\mathrm{gen}}(2,2)$: 249, 277, 295, 349, 353, 388, 389, 394, ...
5. $s_{\mathbb{Q}}^{\mathrm{gen}}(1,1,1,1)$: 61, 73, 79, ...

The small conductor problem is to go as far as possible towards identifying initial segments of sequences $s_E^{\mathrm{cond}}(h)$. When h is at all complicated, at present this often means exhibiting motives that seem to have small conductor for their setting.

Bunched vs. spread out Hodge vectors

Algebraic geometry easily gives many motives M for *particular unimodal* h:

Degree	Curves in $\mathbb{P}^2$	Surfaces in $\mathbb{P}^3$	Three-folds in $\mathbb{P}^4$
3	$(1,1)$	$(0,6,0)$	$(0,5,5,0)$
4	$(3,3)$	$(1,19,1)$	$(0,30,30,0)$
5	$(6,6)$	$(4,44,4)$	$(1,101,101,1)$
6	$(10,10)$	$(10,85,10)$	$(5,255,255,5)$

Automorphic forms most directly contribute to sequences with *spread out* Hodge vectors, $(1,0,0,0,1,0,0,1,0,0,0,1)$.

We have been spoiled by $h=(1,1)$, where both approaches work well!

In general, we know very little about the sequences $s(h)$. E.g. is the sequence $s^{\mathrm{irred}}(1,0,0,2,3,3,2,0,0,1)$ non-empty?

2. The explicit formula

In this section, we sketch the Guinand-Weil explicit formula as it appears in Mestre's 1988 *Compositio* paper *Formules explicites et minorations de conducteurs de variétés algébriques*.

Throughout, we assume the Riemann hypothesis for all L-functions. Without this assumption, the final lower bound obtained is considerably weaker.

Mestre emphasizes the Hodge vectors (g, g) for abelian varieties and $(1, 0, \ldots, 0, 1)$ for modular forms. We emphasize here that one gets non-trivial lower bounds for quite general Hodge vectors h.

The explicit formula

For any motive M with real coefficients and an entire L-function, and any allowed test function F, the Hodge vector h, the conductor N, the analytic rank r, the Frobenius traces $c_{p^e} = \mathrm{Tr}(\mathrm{Fr}_p^e | M)$, and the critical $1/2 + \gamma_k i$ in the upper half plane are related by

$$\begin{aligned} \log N &= 2\pi r + 4\pi \sum_k \hat{F}(\rho_k) + 2\int_0^\infty \hat{F}(t) \sum_j h^j E_j(t)dt \\ &\quad + 2\sum_p \sum_e c_{p^e} \frac{\log p}{p^{e/2}} F\left(\frac{\log p}{2\pi}\right). \end{aligned}$$

Today we are thinking of this explicit formula as an infinite family of exact formulas for $\log N$ which can be used to get lower bounds on $\log N$. There are many other useful perspectives as well!

The Fourier transform and test functions

We require $F(x)$ to be even, compactly supported with $F(0) = 1$, and have two continuous derivatives. Its Fourier transform is then

$$\hat{F}(t) = \int_{-\infty}^{\infty} F(x) e^{-2\pi i t x} dx.$$

Among many standard properities is the scaling property: the Fourier transform of $F(x/z)$ is $z\hat{F}(zt)$.

In this talk, we used only scaled versions of the Odlyzko function:

$$F_{\mathrm{Od}}(x) = \chi_{[-1,1]} \left((1 - |x|) \cos(\pi x) + \frac{\sin |\pi x|}{\pi} \right), \quad (\text{in } C^2 \text{ but not } C^3).$$

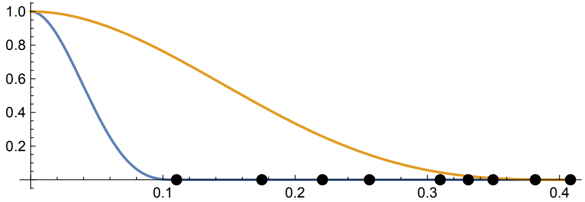
Its Fourier transform is

$$\hat{F}_{\mathrm{Od}}(t) = \frac{8 \cos^2(\pi t)}{\pi^2 (1 - 4t^2)^2} \quad (\text{quartic decay at } \infty).$$

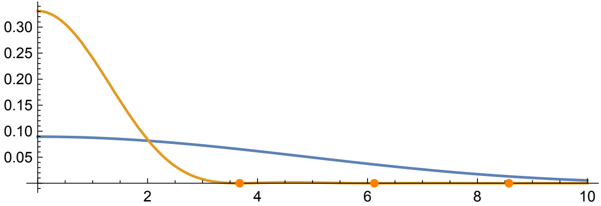
For scaling we use $F_z(x) = F_{\mathrm{Od}}(2\pi x / \log z)$

Plots of typical test functions

One would like both F and $\hat{F}$ to very localized, but this is impossible because of the uncertainty principle. F_2 and F_{13}:



$\widehat{F}_2$ and $\widehat{F}_{13}$:



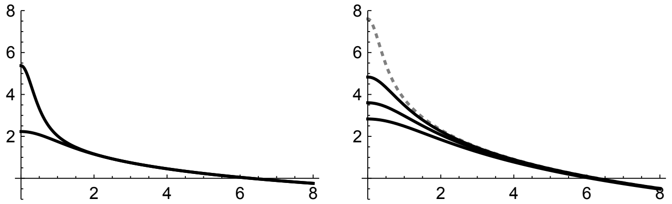
Zero density functions

$h^j E_j(t)$ is "the negative of the part of the expected zero density due to the Hodge number h^j". Using $\psi(s) = \Gamma'(s)/\Gamma(s)$,

$$E_{0+}(t) = \log \pi - \mathrm{Re}\left(\psi\left(\frac{1}{4} + \frac{it}{2}\right)\right), \qquad E_j(t) = 2\log 2\pi - 2\mathrm{Re}\left(\psi\left(\frac{1}{2} + \frac{j}{2} + it\right)\right),$$

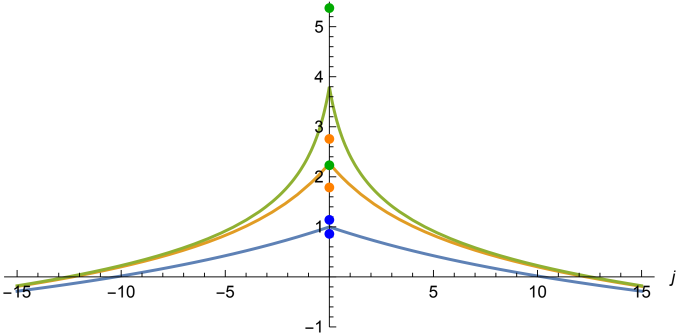
$$E_{0-}(t) = \log \pi - \mathrm{Re}\left(\psi\left(\frac{3}{4} + \frac{it}{2}\right)\right).$$

Graphs of $E_{0+}(t)$ over $E_{0-}(t)$ on the left and $(E_0(t))$, $E_1(t)$, $E_2(t)$, and $E_3(t)$ on the right:



Combining the test and density functions

The quantity $\nu_z(j) = 2\int_0^\infty \widehat{F}_z(t)E_j(t)dt$ appears in the explicit formula. Graphs of $\nu_2(j)$, $\nu_{13}(j)$, and $\nu_\infty(j)$:



One has

$$\begin{aligned} \nu_\infty(0) &= \log(8\pi e^\gamma) &\approx \log(44.76) &\approx 3.80, \\ \nu_\infty(0+) &= \log(8\pi e^{\gamma+\frac{\pi}{2}}) &\approx \log(215.33) &\approx 5.37. \end{aligned}$$

A general lower bound on conductors

Theorem. *Let M be a weight w motive with $L(s, M)$ entire and satisfying the Riemann hypothesis.*

Let $h = (h^{-w}, h^{2-w}, \ldots, h^{w-2}, h^{w})$ be its analytically normalized Hodge vector and let N be its conductor.

Let z be such that

$$c_q \geq 0 \text{ for } q < z.$$

Define $N_{z,h} = \exp\left(\sum_j h^j \nu_z(j)\right)$. Then

$$N > N_{z,h}.$$

The bound for $z = 2$ applies to all motives, the stronger bound for $z = 3$ applies to "half" the motives, etc.

Comparison with modular forms

In the rank two case, the sequence of conductors is known via modular forms. The first conductor N_w coming from the first motive M_w is only slightly more than the 2-bound from the theorem.

w	h_w	bound N_{2,h_w}	N_w	form for M_w
1	$(1,1)$	5.64	11	$(\eta(z)\eta(11z))^2$
2	$(1,0,0,1)$	3.50	5	$(\eta(z)\eta(5z))^4$
3	$(1,0,0,0,0,1)$	2.32	3	$(\eta(z)\eta(3z))^6$
7	$(1,0,0,0,0,0,0,1)$	1.63	2	$(\eta(z)\eta(2z))^8$
9	$(1,0,0,0,0,0,0,0,0,1)$	1.19	2	
11	$(1,0,0,0,0,0,0,0,0,0,0,1)$	0.90	1	$\eta(z)^{24}$

Using Serre's improvement of the Hasse-Weil bound on c_p, Mestre increased z from 2 to 3.78 and improved 5.64 to 10.32.

Comparison with a theorem of Zak

A recent theorem of Zak says that, in a very broad range, Hodge vectors of the middle cohomology of $(n-1)$-dimensional varieties are asymptotically proportional to the n^{th} row of the Eulerian triangle:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 4 & & 1 & & & \\
 & & 1 & & 11 & & 11 & & 1 & & \\
 & 1 & & 26 & & 66 & & 26 & & 1 & \\
1 & & 57 & & 302 & & 302 & & 57 & & 1
\end{array}$$

As a consequence the h^j are roughly Gaussian with standard deviation $\sqrt{(n+1)/12}$. This makes the Hodge numbers bunched enough that $N_{2,h} > 1$ for n into the hundreds.

The “marker” $N_{\infty,h}$

The theorem applies to *reducible* motives such as $M = \sum_{j=1}^{11} h^j M_j$, where it can be fairly sharp, even for $z = 2$.

However for *irreducible* motives, the c_p, which always have mean zero, *also have standard deviation* 1.

For large degree d, this is *much less* variation than allowed by the Hasse-Weil bound $|c_q| \le d$.

Accordingly, for large degree d we expect that it is quite rare for N to be less than $N_{\infty,h}$. Thus, especially for large degree d, $N_{\infty,h}$ provides some guidance when working with the sequence $s(h)$.

3. Special HGMs with small conductor

One has many one-parameter families of hypergeometric motives $H(A, B, t)$, covering e.g. all Hodge vectors in degree < 20.

One cannot expect these families to contain anywhere near all motives for a given h, since the motives $H(A, B, t)$ tend very strongly to be *very wildly* ramified at small primes. Special HGMs, i.e. those with $t = 1$, are *only* ramified at small primes.

On the next three slides, we consider six odd-weight h and look at the smallest conductors arising from special hypergeometric motives with motivic Galois group all of GSp_d. Currently, for a given h, it is hard to find motives with smaller conductor.

All analytic ranks are 0 and 1, and those with 1 are have conductors in italics. There are special HGMs with apparent analytic rank 2 and 3, but their conductors are larger.

Degree two special HGMs with small conductor

$h=(1,1)$		
N	a	b
5.6	**2-bound**	
11	**actual $\mathbb{R}$-lowest**	
24	[2, 2, 6]	[1, 1, 3]
48	[4, 6]	[1, 1, 3]
50	[10]	[1, 1, 2, 2]
50	[5]	[1, 1, 4]
54	[3, 6]	[1, 1, 2, 2]
54	[6, 6]	[1, 1, 2, 2]
72	[2, 2, 6]	[1, 1, 4]
75	[5]	[1, 1, 3]
96	[4, 4]	[1, 1, 3]
125.2	**∞-marker**	
128	[4, 4]	[8]

$h=(1,0,0,1)$		
N	a	b
3.5	**2-bound**	
5	**actual $\mathbb{R}$-lowest**	
6	[2, 2, 3]	[1, 1, 6]
8	[2, 2, 2, 2]	[1, 1, 4]
12	[2, 2, 2, 2]	[1, 1, 6]
16	[2, 2, 4]	[1, 1, 1, 1]
16.9	**∞-marker**	
18	[2, 2, 3]	[1, 1, 4]
24	[2, 2, 2, 2]	[1, 1, 3]
25	[2, 2, 2, 2]	[10]
25	[1, 1, 1, 1]	[5]
27	[3, 3]	[1, 1, 1, 1]
32	[4, 4]	[1, 1, 1, 1]

Note that we are getting many of the small $N = 2^a 3^b 5^c$.

Mobile degree four special HGMs with small N

$h = (2,2)$		
N	a	b
31.8	**2-bound**	
249	**actual $\mathbb{Q}$-lowest**	
1536	$[2,2,4,6]$	$[3,8]$
2592	$[2,2,6,6]$	$[1,1,4,4]$
6144	$[6,8]$	$[1,1,3,4]$
10368	$[2,2,4,4]$	$[3,8]$
10368	$[3,8]$	$[1,1,2,2,4]$
10368	$[6,8]$	$[1,1,4,4]$
15552	$[2,2,4,4]$	$[3,3,6]$
15552	$[3,6,6]$	$[1,1,4,4]$
15683.6	**∞-marker**	

$h = (1,1,1,1)$		
N	a	b
19.7	**2-bound**	
61	**actual lowest**	
96	$[4,4,6]$	$[1,1,1,1,3]$
128	$[2,2,8]$	$[1,1,4,4]$
384	$[4,4,6]$	$[3,8]$
384	$[3,4,4]$	$[1,1,2,2,6]$
384	$[2,2,2,2,6]$	$[1,1,3,4]$
384	$[3,8]$	$[1,1,4,6]$
486	$[3,3,6]$	$[1,1,1,1,2,2]$
768	$[2,2,4,6]$	$[1,1,1,1,3]$
2122.5	**∞-marker**	

Existing complete tables for these h are dominated by conductors involving large primes, which don't arise as special HGMs.

Rigid degree four special HGMs with small N

$h=(2,0,0,2)$		
N	a	b
12.2	**2-bound**	
256	$[2,2,2,2,4]$	$[1,1,8]$
287.2	**∞-marker**	
384	$[3,8]$	$[1,1,1,6]$
1944	$[3,3,6]$	$[1,1,1,1,4]$
2048	$[2,2,8]$	$[1,1,1,1,4]$
2592	$[2,2,3,6]$	$[1,1,1,1,4]$
2592	$[2,2,2,2,6]$	$[1,1,12]$
5000	$[2,2,5]$	$[1,1,1,1,4]$
5184	$[2,2,2,2,3]$	$[1,1,12]$
6912	$[2,2,8]$	$[1,1,1,1,6]$

$h=(1,1,0,0,1,1)$		
N	a	b
8.1	**2-bound**	
32	$[2,2,2,2,2,2]$	$[1,1,4,4]$
48	$[2,2,3,4]$	$[1,1,1,1,6]$
96	$[2,2,2,2,3]$	$[1,1,4,6]$
105.6	**∞-marker**	
128	$[2,2,2,2,2,2]$	$[1,1,1,1,4]$
162	$[2,2,2,3]$	$[1,1,6,6]$
243	$[6,6,6]$	$[2,2,2,2,2,2]$
243	$[3,3,3]$	$[1,1,1,1,1,1]$
256	$[2,2,2,2,4]$	$[1,1,1,1,1,1]$
256	$[2,2,8]$	$[1,1,1,1,1,1]$

Note that $(2,0,0,2)$ is hard to make from either algebraic geometry or automorphic representations, and this is perhaps the "source" of the large conductors.

4. A factorization $M = M_6 \oplus M_8$

The explicit formula lets me numerically resolve two questions I asked in my October 19 lecture. Here are the relevant parts of that lecture.

The motive $M = H([2^{16}],[1^{16}],1)$ has Hodge vector $(1,1,1,1,1,1,1,0,0,1,1,1,1,1,1,1)$, a certified-to-10-digits $\Lambda(M,s)$, with conductor 2^{15}, sign 1, order of central vanishing 2, and $L''(M,8) \approx 7.851654518$.

The first two Frobenius polynomials (two seconds and thirty seconds):

$$
\begin{aligned}
F_3(x) \quad = \quad & (1 - 268 \cdot 3x + 204193 \cdot 3^4 x^2 - 1001800 \cdot 3^9 x^3 + 204193 \cdot 3^{19} x^4 - 268 \cdot 3^{31} x^5 + 3^{45} x^6) \\
& (1 + 2992 \cdot x + 39116 \cdot 3^4 x^2 - 7596496 \cdot 3^6 x^3 - 203836426 \cdot 3^{12} x^4 \\
& \qquad -7596496 \cdot 3^{21} x^5 + 39116 \cdot 3^{34} x^6 + 2992 \cdot 3^{45} x^7 + 3^{60} x^8)
\end{aligned}
$$

$$
\begin{aligned}
F_5(x) \quad = \quad & (1 + 1614 \cdot 5^3 x + 28284579 \cdot 5^4 x^2 + 1394686516 \cdot 5^9 x^3 + 28284579 \cdot 5^{19} x^4 + 1614 \cdot 5^{33} x^5 + 5^{45} x^6) \\
& (1 - 41208 \cdot x - 44999364 \cdot 5^3 x^2 - 22376708712 \cdot 5^6 x^3 + 3926679014806 \cdot 5^{12} x^4 \\
& \qquad -22376708712 \cdot 5^{21} x^5 - 44999364 \cdot 5^{33} x^6 - 41208 \cdot 5^{45} x^7 + 5^{60} x^8)
\end{aligned}
$$

Two questions from October 19

The splitting $M = M_6 \oplus M_8$ is known *a priori* from a joint symmetry $t \leftrightarrow 1/t$ and $2 \leftrightarrow 1$. The Hodge vectors of the summands are

$$\begin{aligned} h_6 &= (0,1,0,1,0,1,0,0,0,0,1,0,1,0,1,0), \\ h_8 &= (1,0,1,0,1,0,1,0,0,1,0,1,0,1,0,1). \end{aligned}$$

The two Frobenius polynomials suffice to prove that the motivic Galois group of the two factors are GSp_6 and GSp_8.

Q1. Since $L_2(M,s) = 1$, there are only two possibilities for $(\mathrm{cond}(M_6), \mathrm{cond}(M_8))$, namely $(2^6, 2^9)$ or $(2^7, 2^8)$. Which is it?

Q2. There are only three possibilities for $(\mathrm{rank}(M_6), \mathrm{rank}(M_8))$, namely $(2,0)$, $(1,1)$, or $(0,2)$. Which one is correct?

Calculating and factoring a few $F_p(x)$

We have tons of c_{p^e}. However, to get the decomposition $c_{p^e} = c^6_{p^e} + c^8_{p^e}$, even for just $e = 1$, we need to factor all of $F_p(x)$. The next two (8 minutes and 2.5 hours):

$$
\begin{aligned}
F_7(x) &= \Big(1 + 248232 \cdot 7x + 36864645 \cdot 7^4 x^2 - 12114440144 \cdot 7^9 x^3 + 36864645 \cdot 7^{19} x^4 + \\
&\qquad 248232 \cdot 7^{31} x^5 + 7^{45} x^6\Big) \cdot \\
&\quad \Big(1 + 667104x + 92084011804 \cdot 7^2 x^2 + 107704347009888 \cdot 7^6 x^3 + 216772203079210 \cdot 7^{13} x^4 \\
&\qquad + 107704347009888 \cdot 7^{21} x^5 + 92084011804 \cdot 7^{32} x^6 + 667104 \cdot 7^{45} x^7 + 7^{60} x^8\Big)
\end{aligned}
$$

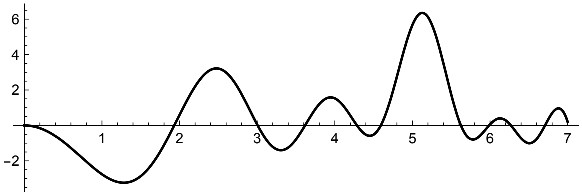
$$
\begin{aligned}
F_{11}(x) &= \Big(1 - 883812 \cdot 11x + 86399921193 \cdot 11^4 x^2 - 113266524342552 \cdot 11^9 x^3 + 86399921193 \cdot 11^{19} x^4 \\
&\qquad - 883812 \cdot 11^{31} x^5 + 11^{45} x^6\Big) \\
&= \Big(1 + 34438544x + 7563161639884 \cdot 11^2 x^2 - 5931371880123984 \cdot 11^7 x^3 + 1164681420132811670 \cdot 11^{12} x^4 \\
&\qquad - 5931371880123984 \cdot 11^{22} x^5 + 7563161639884 \cdot 11^{32} x^6 + 34438544 \cdot 11^{45} x^7 + 11^{60} x^8\Big)
\end{aligned}
$$

Getting a few zeros of $\Lambda(s, M)$

Calculating $\Lambda(s, M)$ with enough precision to make each contribution to $\log N$ very likely accurate to five decimal places gives

$$\rho_1 \approx 1.93195000805, \quad \rho_2 \approx 3.00559765, \quad \rho_3 \approx 3.61679, \quad \ldots$$

The Hardy Z-function on $[0, 7]$ is



Note that this calculation does not give any hints as to the desired factorization $Z(t) = Z_6(t)Z_8(t)$. In other words, we do not know which motive a given t_i belongs to.

Applying the explicit formula to M_6 and M_8

Plugging into the explicit formula, dividing all terms by $\log 2$ for greater clarity, and keeping track of partial sums:

	term_6	(Tends to 6 or 7) total_6	term_8	(Tends to 8 or 9) total_8	Comments
h	3.11324	3.11324	4.86171	4.86171	
3	0.17011	3.28335	−0.63306	4.22866	
5	−0.35472	2.92864	0.07245	4.30111	
7	−0.07386	2.85477	−0.02836	4.27275	
9	−0.02269	2.83209	0.00183	4.27458	
11	0.00028	2.83237	−0.00101	4.27357	
r	2.99946	5.83183	2.99946	7.27303	Forced! $A2 = (1, 1)$
ρ_1		5.83183	1.68061	8.95364	Forced! $A1 = (2^6, 2^9)$
ρ_2	0.13610	5.96793		8.95364	Forced!
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
Total		6.00000		9.00000	

New challenges

Because their Hodge vectors are so spread, the conductors are actually not that small. Namely, the bound $N_{2,h_6} \approx 1.96$ and even the marker $N_{\infty,h_6} \approx 11.29$ are much less than $2^6 = 64$, while $N_{2,h_8} \approx 2.91$ and $N_{\infty,h_8} \approx 29.4$ are likewise much less than $2^9 = 512$.

Problem 1. *Find motives which have these Hodge numbers but smaller conductor.*

Problem 2. *Improve the general hypergeometric theory so that one can calculate directly on the summands of* $M([2^d],[1^d],\pm 1)$. Then one could work analytically for d up through around 30, explicitly seeing factorizations like $M = M_{13} \oplus M_{15}$.