DESIGNER K3 SURFACES

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We will focus our attention on a limited part of the story. Our goal: how do we produce K3 surfaces (over $\mathbb{Q}$, or a number field) with certain specifications?

By specifications, we might want them for purposes of modularity or L-functions, etc.

1. Periods of K3 surfaces

We begin with the story over $\mathbb{C}$. Let X be a complex projective K3 surface: a smooth projective surface, simply connected ($\pi_1(X) = \{1\}$) and $K_x = -c_1(T_X) = 0$. Pick ω a holomorphic 2-form realizing a trivialization, so $\omega \in \Gamma(\Omega^2_X)$; this gives you an isomorphism $\mathcal{O} \xrightarrow{\sim} \Omega^2_X$. So a K3 surface admits a holomorphic symplectic structure. The form ω is closed, $d\omega = 0$, then $\omega \in H^2(X, \mathbb{C})$.

Since $\pi_1(X) = \{1\}$, we have $\Gamma(\Omega^1_X) = 0$. Therefore, the Hodge numbers (using duality) are

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & * & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

In particular, $H^1(\mathcal{O}_X) = 0$ and $H^2(\Omega^1_X) = H^1(\Omega^2_X) = 0$. The mystery number is determined by Riemann–Roch:

$$2 = \chi(\mathcal{O}_X) = \frac{c_1(X)^2 + c_2(X)}{12}$$

and $c_1(X) = 0$ so $c_2 = 24$, so the mystery number is $* + 4 = 24$ has $* = 20$.

Let $\Lambda = H^2(X, \mathbb{Z})$ as a lattice with respect to the intersection form $(,)$. The signature of this lattice is $(3, 19)$ from the Hodge numbers; it is unimodular and even, from the adjunction formula (liar's proof: if D is a curve, then $D \cdot D = 2g - 2$ is even). This information about the lattice implies that

$$\Lambda = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 3 \oplus (-E_8)^{\oplus 2}.$$

Fix a basis $e_1, \ldots, e_{22}$ of $\Lambda^* = H_2(X, \mathbb{Z})$. Then the periods of X are

$$\omega_j = \int_{e_j} \omega$$

for $j = 1, \ldots, 22$. Another way to think about this is to take $\omega = \sum_{j=1}^{22} \omega_j e^j$ where e^j is the dual basis in Λ.

The form is only determined up to scalar, so $[\omega] \in \mathbb{P}^1(\Lambda \otimes \mathbb{C})$, so

$$[\omega] \in \mathcal{D}_\Lambda \subset \mathbb{P}(\Lambda \otimes \mathbb{C})$$

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where

$$\mathcal{D}_\Lambda = \{\tau \in \Lambda \otimes \mathbb{C} : (\tau, \tau) = 0 \text{ and } (\tau, \overline{\tau}) > 0\}.$$

This follows from the fact that the square of a form cannot exist by hypothesis, and the second is an orientation condition that we suppress. We call this $\tau(X) = [\omega]$. If we vary the basis, $\mathrm{Aut}(\Lambda)$ act on $\mathcal{D}_\Lambda$ via change of basis.

2. Néron–Severi groups

To achieve our goal, we need to now turn to a more precise analysis of how the periods relate to the desired specifications. We begin with the Néron–Severi groups via periods. The Lefschetz $(1,1)$-theorem states that

$$\begin{aligned}\mathrm{NS}(X) &= H^2(X, \mathbb{Z}) \cap H^1(\Omega^1_X) \\ &= \{D \in H^2(X, \mathbb{Z}) : (D, \omega) = 0\}.\end{aligned}$$

The hyperplane class is nontrivial in $\mathrm{NS}(X)$.

Example 2.1. Suppose $\mathrm{NS}(X) = \mathbb{Z}h$, h a polarization (a choice of hyperplane class). We consider $h^\perp \subseteq \Lambda$. If $(h, h) = 2g - 2$ with $g \geq 2$, then $\Lambda_g = h^\perp$ has rank 21 and looks like

$$\Lambda_g \simeq (2 - 2g) \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus 2} \oplus (-E_8)^{\oplus 2}.$$

We have $(\omega, h) = 0$, so

$$\mathbb{P}(\Lambda_g \otimes \mathbb{C}) \subset \mathbb{P}(\Lambda \otimes \mathbb{C})$$

and inside each we have

$$\mathcal{D}_{\Lambda_g} = \{\tau : (\tau, \tau) = 0, (\tau, \overline{\tau}) > 0\} \subset \mathcal{D}_\Lambda$$

and $\tau(X, h) \in \mathcal{D}_{\Lambda_g}$. The fact that the remaining part of lattice is orthogonal to the hyperplane reduces the side of the lattice and correspondingly constrains the periods.

What are "natural" discrete groups acting on $\mathcal{D}_{\Lambda_g}$? We might take

$$\mathrm{Aut}(\Lambda, h) = \{\alpha \in \mathrm{Aut}(\Lambda) : \alpha(h) = h\}.$$

(You might allow h to change sign.) We could also take $\mathrm{Aut}(\Lambda_g)$ but not all lift to Λ. So we should remain flexible depending on the application we have in mind. (The latter gives isogeny classes, coming from a bigger group.)

Example 2.2. Let $N \subset \Lambda$ be a saturated sublattice. Hodge tells us a constraint on the signature: $\mathrm{sgn}(N) = (1, m)$. Let $N^\perp \subset \Lambda$ be the complement. Then we can play the same game as before: we look at

$$\mathbb{P}(N^\perp \otimes \mathbb{C}) \subset \mathbb{P}(\Lambda \otimes \mathbb{C})$$

and the corresponding period domains

$$\mathcal{D}_{N^\perp} \subset D_\Lambda$$

taking a linear slice as in the previous example. In this case, we have discrete groups

$$\{\gamma \in \mathrm{Aut}(\Lambda) : \gamma|_N = 1\}$$

or

$$\{\gamma \in \mathrm{Aut}(\Lambda) : \gamma(N) \subseteq N\}$$

or just $\mathrm{Aut}(N^\perp)$.

In the second case, the automorphisms might not act faithfully. For example, let $N = \begin{pmatrix} 2 & 0 \\ 0 & -2d \end{pmatrix}$; solving Pell's equation gives us a cyclic group of automorphisms, some cyclic subgroup will lift to automorphisms of the lattice, but we do not see them on the level of periods.

3. Torelli Theorems

(1) K3 surfaces are determined by the periods. Suppose X_1 and X_2 are K3 surfaces admitting an isometry (isomorphism of lattices)

$$\phi : H^2(X_1, \mathbb{Z}) \xrightarrow{\sim} H^2(X_2, \mathbb{Z})$$

with $\phi([\omega_1]) = [\omega_2]$. Then $X_1 \simeq X_2$ (but this isomorphism need not necessarily induce ϕ).

(2) All periods arise: every $\tau \in \mathcal{D}_\Lambda$ arises from a K3 surface. More precisely, every $\tau \in \mathcal{D}_{N^\perp}$ as in the example, arises from a K3 surface X admitting a sublattice $N \subseteq \mathrm{NS}(X)$.

(3) Let $\mathcal{M}_N$ be a moduli space of K3 surfaces with a sublattice $N \subseteq \mathrm{NS}(X)$ with $h \in \mathrm{NS}(X)$ a polarization. Then the period map τ maps

$$\tau : \mathcal{M}_N \to \Gamma\backslash\mathcal{D}_{N^\perp}$$

where Γ is an arithmetic group, as in the examples. Both sides of this map have the same dimension; the quotient $\Gamma\backslash\mathcal{D}_{N^\perp}$ is quasiprojective (Baily–Borel, a Type IV Shimura variety); the map τ is algebraic (using hyperbolicity results of Borel). As a result, both sides (and the map τ) can be defined over a number field; in many cases, we know this number field explicitly.

A dimension count: we specified the signature $\mathrm{sgn}(N) = (1, m)$. So the dimension over $\mathbb{C}$ of $\mathcal{D}_N$ is $20 - (1 + m) = 19 - m$; as N gets bigger, the dimension gets smaller.

4. Geometry from the periods

Let (X, h) be a polarized K3 surface. We can read off the following from $\tau(X) = [\omega]$.

- $\mathrm{NS}(X) \in h$.
- Effective curves $\mathrm{Eff}(X)$, which elements of the Néron–Severi group are defined by actual curves: it is the monoid generated by (Riemann–Roch)

$$\{D \in \mathrm{NS}(X) : D^2 = (D, D) \geq -2, (D, h) > 0\}.$$

- All of the smooth rational curves: we $R = [\mathbb{P}^1]$ if and only if $R \in \mathrm{Eff}(X)$ is indecomposable and $R^2 = -2$.
- Elliptic fibrations: X is elliptic if and only if $\mathrm{NS}(X)$ represents 0 (in the sense of quadratic forms).
- Automorphisms:

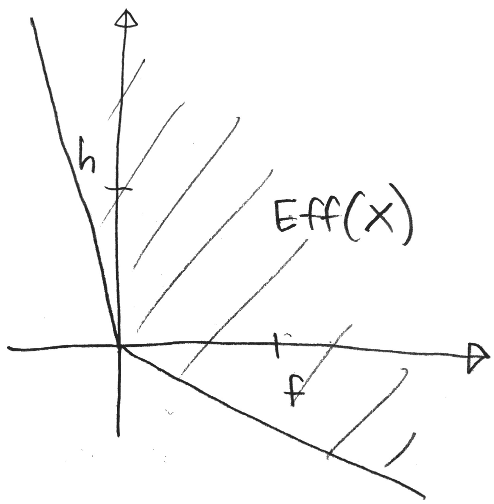
$$\mathrm{Aut}(X) = \{\rho \in \mathrm{Aut}(\Lambda) : \rho^*([\omega]) = [\omega] \text{ and } \rho^* \mathrm{Eff}(X) = \mathrm{Eff}(X)\}.$$

Example 4.1. If $X \supset R \simeq \mathbb{P}^1$, we can define a reflection

$$\rho_R : H^2(X,\mathbb{Z}) \xrightarrow{\sim} H^2(X,\mathbb{Z})$$
$$\gamma \mapsto \gamma + (\gamma, R)R$$

This reflection is an automorphism of the Hodge structure but is *not* an automorphism of X. (The reflection messes with $\mathrm{Eff}(X)$.)

Example 4.2. Let $N = \mathbb{Z}h + \mathbb{Z}f$ be the lattice with intersection pairing $\begin{pmatrix} 2 & 5 \\ 5 & 2 \end{pmatrix}$. We have h ample, it does not represent $0, -2$. We have $\mathrm{NS}(X) = N$, $\mathrm{Aut}(X) = \langle \rho_1, \rho_2 \rangle$ with $\rho_1(\gamma) = -\gamma + (\gamma, h)h$ and $\rho_2(\gamma) = -\gamma + (\gamma, f)f$.



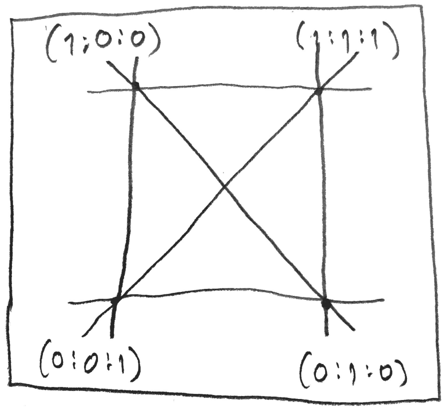
Homework problem: Let $X = \{F_{12} = F_{21} = 0\} \subseteq \mathbb{P}^2 \times \mathbb{P}^2$, cut out by equations of bidegree $(1,2)$ and $(2,1)$.

5. Arithmetic examples with lots of structure

We have $\mathrm{rk}(\mathrm{NS}(X)) \leq 20$. When there is equality, the period domain is zero-dimensional. (By the Torelli theorem, such a K3 surface is defined over a number field.)

Suppose that X is defined over $\mathbb{Q}$ and $\mathrm{NS}(X) \simeq \mathbb{Z}^{20}$ and all generators are defined over $\mathbb{Q}$. This is so restrictive that maybe X does not even exist. (Even the Fermat quartic $w^4 + x^4 = y^4 + z^4$ fails.) We have $T(X) = \mathrm{NS}(X)^\perp \subset \Lambda$ is a rank 2 lattice, positive definite. There is an equivalence between rank 20 K3 surfaces over $\mathbb{C}$ and oriented even positive definite rank two lattices (this goes back to Shafarevich and Shioda). But the other conditions imply that $T(X)$ is primitive with class number 1. There are exactly 13 of these!

Example 5.1. Take the double cover of $\mathbb{P}^2$ branched over the following configuration:



This is given by the desingularization.

$$z^2 = xy(x-1)(y-1)(x-y)$$

(there is also branching at infinity).

The summary (Elkies–Schuett):

(1) For each of the 13 examples, there exists an elliptic K3 surface X over $\mathbb{Q}$ with $\mathrm{NS}(X) = \mathbb{Z}^{20}$ and $T(X)$ as specified.
(2) You can classify the $\mathbb{Q}$-isomorphism types.
(3) (Livné) One can prove modularity of the L-functions.

As a final application, let X be a K3 surface with maximal nonsymplectic automorphisms. Most of these are defined over $\mathbb{Q}$, and you can prove the modularity of their L-function, basically because of the CM structure given by the automorphism.