Semistable reduction of curves and computation of bad Euler factors of L-functions

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preliminary version, comments welcome

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1 Introduction

Let Y be a smooth projective curve of genus g defined over a number field K. The *L-function* of Y is a Euler product

$$L(Y,s) := \prod_{\mathfrak{p}} L_{\mathfrak{p}}(Y,s),$$

where $\mathfrak{p}$ ranges over the prime ideals of K. The local L-factor $L_{\mathfrak{p}}(Y,s)$ is defined as follows. Choose a decomposition group $D_{\mathfrak{p}} \subset \mathrm{Gal}(K^{\mathrm{abs}}/K)$ of $\mathfrak{p}$. Let $I_{\mathfrak{p}} \subset D_{\mathfrak{p}}$ be the inertia subgroup and let $\sigma_{\mathfrak{p}} \in D_{\mathfrak{p}}$ an arithmetic Frobenius element (i.e. $\sigma_{\mathfrak{p}}(\alpha) \equiv \alpha^{\mathrm{N}\mathfrak{p}} \pmod{\mathfrak{p}}$). Then

$$L_{\mathfrak{p}}(Y,s) := \det\big(1 - (\mathrm{N}\mathfrak{p})^{-s}\sigma_{\mathfrak{p}}^{-1}|V^{I_{\mathfrak{p}}}\big)^{-1},$$

where

$$V := H^1_{\mathrm{et}}(Y \otimes_K K^{\mathrm{abs}}, \mathbb{Q}_\ell)$$

is the first étale cohomology group of Y (for some auxiliary prime ℓ distinct from the residue characteristic p of $\mathfrak{p}$). We refer to § 2.2 for more details.)

Another arithmetic invariant of Y closely related to $L(Y,s)$ is the *conductor* of the curve. Similar to $L(Y,s)$, it is a product of local factors (multiplied by a power of the discriminant δ_K of K):

$$N := \delta_K^{2g} \cdot \prod_{\mathfrak{p}} (\mathrm{N}\mathfrak{p})^{f_{\mathfrak{p}}},$$

where $f_{\mathfrak{p}}$ is a nonnegative integer called the *conductor exponent* at $\mathfrak{p}$. The integer $f_{\mathfrak{p}}$ measures the *ramification* of the Galois module V at the prime $\mathfrak{p}$. It is defined as the integer

$$f = f_{Y/K} = \epsilon + \delta, \tag{1.1}$$

where

$$\epsilon := \dim V - \dim V^{I_K} \tag{1.2}$$

is the codimension of the I_K-invariant subspace and δ is the *Swan conductor* of V (see [14] § 2, or [16], § 3.1). We discuss this in more detail in § 2.3.

In these notes we discuss how to compute the local factor $L_{\mathfrak{p}}(Y,s)$ and the conductor exponent $f_{\mathfrak{p}}$ at a prime of bad reduction for *superelliptic curves* (§ 3.1). More precisely, we will show that $L_{\mathfrak{p}}(Y,s)$ and $f_{\mathfrak{p}}$ can easily be computed from the knowledge of the *semistable reduction* of Y at $\mathfrak{p}$. Furthermore, we will explain how to compute semistable reduction explicitly for superelliptic curves. The main reference for this course are the three papers [3], [1] and [12].

The local L-factor at the bad primes may alternatively also be computed using a regular model, In the special case of elliptic curves the conductor exponent $f_{\mathfrak{p}}$ may be computed using Ogg's formula ([15]). As far as we know there is no general method for computing the conductor exponent at the bad primes from the regular model in general. These notes we calculate the local invariants $L_{\mathfrak{p}}$ and $f_{\mathfrak{p}}$ of several concrete examples, including several elliptic curves. We hope that this facilitates the comparison between the two methods.

2 Semistable reduction

In this section we give some background on stable reduction and discuss how the local L-factor and the conductor exponent may be computed using the stable reduction.

Since the L-factor $L_{\mathfrak{p}}$ and the conductor exponent $f_{\mathfrak{p}}$ are local invariants, we assume from now one that K is a finite extension of $\mathbb{Q}_p$ for some fixed prime number p. The residue field of K is a finite field, which we denote by $\mathbb{F}_K$. We write $\mathcal{O}_K$ for the ring of integers, π_K for the uniformizing element, and $\mathfrak{m}_K$ for the maximal ideal of K. Let $v_K : K \to \mathbb{Q} \cup \{\infty\}$ be the p-adic valuation of K which is normalized such that $v_K(p) = 1$. For a finite extension L/K we use the symbols $\mathcal{O}_L, \mathbb{F}_L$, and π_L analogously. Choose an algebraic closure $K^{\rm alg}$ of K and write $\Gamma_K = \mathrm{Gal}(K^{\rm alg}/K)$ for the absolute Galois group of K. The residue field of $K^{\rm alg}$ is denoted by k; it is the algebraic closure of $\mathbb{F}_K$.

2.1 Definitions

Let Y be a smooth projective absolutely irreducible curve over K. A *model* of Y is a flat proper normal $\mathcal{O}_K$-scheme $\mathcal{Y}$ with generic fiber $\mathcal{Y} \otimes_{\mathcal{O}_K} K \simeq Y$ isomorphic to Y. We denote the special fiber of $\mathcal{Y}$ by $\bar{Y}$ by $\mathcal{Y}_s$. If the model $\mathcal{Y}$ is clear from the context we write $\bar{Y}$ instead of $\mathcal{Y}_s$.

Definition 2.1 (a) A curve Y over K has *good reduction* if there exists a smooth model of Y. Otherwise we say that Y has *bad reduction*.

(b) A curve Y over K has *potentially good reduction* if there exists a finite extension L/K such that $Y_L := Y \otimes_K L$ has good reduction.

(c) A curve Y over K has *semistable reduction* if there exists a model $\mathcal{Y}$ of Y whose special fiber $\bar{Y}$ is semistable, i.e. is reduced and has at most ordinary double points as singularities.

If Y has genus 0, then Y always has potentially good reduction, and has good reduction if and only if it has a rational point. From now on we assume that $g(Y) \geq 1$. Note that potentially good, but not good, reduction is considered as bad reduction according to this definition (Example 2.4).

Let $\phi : Y \to X$ be a cover of smooth projective curves over K. It is uniquely determined by the extension of function fields $K(Y)/K(X)$. For a model $\mathcal{X}$ of X the normalization $\mathcal{Y}$ of $\mathcal{X}$ inside $K(Y)$ is a model of Y, and ϕ extends to a finite morphism $\mathcal{Y} \to \mathcal{X}$.

The main case we consider in these notes is the case that Y is a superelliptic curve over K birationally given by an equation

$$Y:\ y^n = f(x). \tag{2.1}$$

We discuss this case in more detail in § 3.1. In this situation $X = \mathbb{P}^1_K$ is the projective line with coordinate x and $\phi : Y \to X$ corresponds to the function x. The coordinate x naturally defines a model $\mathcal{X}^{\rm naive} = \mathbb{P}^1_{\mathcal{O}_K}$ of X. We define $\mathcal{Y}^{\rm naive}$ as the normalization of $\mathcal{X}^{\rm naive}$ in the function field $K(Y)$ of Y. We call this model of Y the *naive model*. The following lemma gives necessary conditions for the naive model to defined by the equation (2.1).

Lemma 2.2 *Assume that $f(x) \in \mathcal{O}_K[x]$ and that the leading coefficient of f is a unit in $\mathcal{O}_K$. Moreover, we assume that Y is absolutely irreducible and that the $\mathbb{F}_K$-curve defined by (2.1) is reduced. Then* $\mathrm{Spec}(\mathcal{O}_K[x,y]/(y^n - f)$ *defines an open affine part of $\mathcal{Y}^{\rm naive}$.*

The lemma follows from Serre's criterion for normality, see [9], ???

We discuss two concrete examples in genus 1.

Example 2.3 We consider the elliptic curve E over $\mathbb{Q}$ given by the Weierstrass equation

$$E:\ w^2 + w = x^3 - x^2 =: g(x). \tag{2.2}$$

The curve E is taken from Cremona's list and has conductor 11, discriminant -11, and j-invariant $-2^{12}/11$.

The equation (2.2), considered over $\mathbb{F}_2$, defines a smooth elliptic curve over $\mathbb{F}_2$. Hence E has good reduction at $p = 2$.

To consider what happens at the odd primes, we define $y = 2w + 1$ and $f = 4g + 1$. Rewriting (2.2) yields

$$E:\ y^2 = f(x) = 4x^3 - 4x^2 + 1. \tag{2.3}$$

The polynomial f has discriminant $\Delta(f) = -2^4 \cdot 11$. It follows that E only has bad reduction at $p = 11$. (Of course we could also have seen this immediately from the discriminant of E.) Note that

$$f(x) \equiv 4(x+4)(x+3)^2 \pmod{11}.$$

Therefore equation (2.3) considered over $\mathbb{F}_{11}$ defines a nodal cubic, and E has split multiplicative reduction at $p = 11$.

In this example, the elliptic curve $E/\mathbb{Q}$ has semistable reduction at all primes p. In [1] we discuss a class of hyperelliptic curves of arbitrary genus with the same property.

Example 2.4 As a second example we consider the elliptic curve $E/\mathbb{Q}$ defined by

$$E:\; w^2 + w = x^3 - 7.$$

This elliptic curve has conductor 27, discriminant $2^2 \cdot 3^3 \cdot 7$, and $j = 0$. As in Example 2.3, the elliptic curve has good reduction at $p = 2$.

Defining $y = 2w + 1$ we obtain the alternative equation

$$y^2 = f(x) := 4x^3 - 27.$$

The discriminant of f is $-2^4 \cdot 3^9$, hence E has good reduction for $p \neq 3$. At $p = 3$ the elliptic curve E has potentially good (but not good) reduction.

The smooth model of E at $p = 3$ we describe in Examples 3.6 and 3.9 below is only defined over an extension of $\mathbb{Q}_3$ of degree 12, which illustrates that finding the 'right' model, may be rather involved in general. In general it is not feasable to just resolve the singularities on the special fiber of the naive model by explicit blow-up. The main restriction is that one does not know the field L over which Y acquires stable reduction apriori. (The proof of the Stable Reduction Theorem gives such a field, but this field is much too large to work with in praxis.)

The local L-factor L_3 that we compute in Example 3.9 is trivial. This illustrates that from the point of view of local L-factors potentially good, but not good, reduction should be considered as bad reduction.

Theorem 2.5 (Deligne–Mumford) *([5]) There exists a finite extension L/K such that $Y_L = Y \otimes_K L$ has semistable reduction.*

The semistable model $\mathcal{Y}$ from Theorem 2.5 is not unique. However, if we assume that $g := g(Y) \geq 2$ there is a minimal semistable model $\mathcal{Y}^{\rm stab}$ (w.r.t. dominance of models), called the *stable model* of Y_L. The special fiber $\bar{Y}^{\rm stab}$ of $\mathcal{Y}^{\rm stab}$ is called the *stable reduction* of Y_L. It is a stable curve over the residue field $\mathbb{F}_L$, i.e. it is a semistable curve such that every geometric irreducible component of $\bar{Y}$ of genus zero contains at least 3 singular points.

The stable reduction is uniquely determined by the K-curve Y and the extension L/K. The dependence on L is very mild: if L'/L is a further finite extension then the stable reduction of Y corresponding to the extension L'/K is just the base change of $\bar{Y}^{\rm stab}$ to the residue field of L'. In the case that Y is an elliptic curve it is also possible to define a stable model with the same uniqueness properties: one considers the neutral element of the group law on the elliptic curve as marking.

After replacing L by a suitable finite extension we may and will henceforth assume that L/K is a Galois extension. We also choose an embedding $L \subset K^{\rm alg}$. The uniqueness of the stable model implies that the absolute Galois group Γ_K of K acts naturally on the stable model $\mathcal{Y}^{\rm stab}$ via its finite quotient $\Gamma := \mathrm{Gal}(L/K)$. The action of Γ on $\mathcal{Y}^{\rm stab}$ also induces an action on the stable reduction $\bar{Y}^{\rm stab}$.

To compute the local L-factors of Y it is not necessary to consider the stable model. For our purposes it is more convenient to work with a more general class of semistable models which we call quasi-stable models of Y.

Definition 2.6 A semistable $\mathcal{O}_L$-model $\mathcal{Y}$ of Y_L is called *quasi-stable* if the tautological action of Γ on Y_L extends to an action on $\mathcal{Y}$.

The concrete model of a superelliptic curve Y we construct in the following will be unique in some other way than the stable model. Therefor it will be clear from the definition of the model that it is quasi-stable.

2.2 An expression for $L_{\mathfrak{p}}$ at the bad primes

We use the same notations and assumptions as in § 2.1. In particular, Y is a smooth projective absolutely irreducible curve of genus $g \geq 1$ defined over a finite extension K of $\mathbb{Q}_p$. Let L/K be a finite Galois extension, and $\mathcal{Y}$ a quasi-stable $\mathcal{O}_L$-model of Y_L.

Recall that $I_K \lhd \Gamma_K = \mathrm{Gal}(K^{\rm alg}/K)$ is the inertia group of K. We have a short exact sequence

$$1 \to I_K \to \Gamma_K \to \Gamma_{\mathbb{F}_K} \to 1,$$

where $\Gamma_{\mathbb{F}_K} = \mathrm{Gal}(k/\mathbb{F}_K)$ is the absolute Galois group of $\mathbb{F}_K$. This is the free profinite group of rank one generated by the *Frobenius element* σ_q, defined by $\sigma_q(\alpha) := \alpha^q$, where $q = |\mathbb{F}_K|$.

Fix an auxiliary prime $\ell \neq p$, and write

$$V = H^1_{\rm et}(Y_{K^{\rm alg}}, \mathbb{Q}_\ell) := \big(\varprojlim_n H^1_{\rm et}(Y_{K^{\rm alg}}, \mathbb{Z}/\ell^n)\big) \otimes \mathbb{Q}_\ell$$

for the étale cohomology group.

By definition $\Gamma = \mathrm{Gal}(L/K)$ acts on the quasi-stable model $\mathcal{Y}$, and hence also on its special fiber $\bar{Y}$. This action is semilinear, meaning that the structure map $\bar{Y} \to \mathrm{Spec}\,\mathbb{F}_L$ is Γ-equivariant. Let $I \lhd \Gamma$ denote the inertia subgroup, i.e. the image of I_K in Γ. The inertia group $I \lhd \Gamma$, defined as the image of I_K in Γ, acts $\mathbb{F}_L$-linearly on $\bar{Y}$.

The quotient curve

$$\bar{Z} := \bar{Y}/\Gamma$$

has a natural structure of an $\mathbb{F}_K$-scheme, and as such we have

$$\bar{Z}_{\mathbb{F}_L} := \bar{Z} \otimes_{\mathbb{F}_K} \mathbb{F}_L = \bar{Y}/I.$$

Since the quotient of a semistable curve by a finite group of geometric automorphisms is semistable, it follows that $\bar{Z} \otimes_{\mathbb{F}_K} \mathbb{F}_L$ is a semistable curve over $\mathbb{F}_L$. We conclude that $\bar{Z}$ is a semistable curve over $\mathbb{F}_K$. We denote by $\bar{Z}_k := \bar{Z} \otimes_{\mathbb{F}_K} k$ the base change of $\bar{Z}$ to the algebraic closure k of $\mathbb{F}_K$.

Definition 2.7 The $\mathbb{F}_K$-curve $\bar{Z} = \bar{Y}/\Gamma$ is called the *inertial reduction* of Y, corresponding to the quasi-stable model $\mathcal{Y}$.

The local L-factor is defined as

$$L_{\mathfrak{p}}(Y,s) := \det\left(1-(\mathrm{N}\mathfrak{p})^{-s}\sigma_{\mathfrak{p}}^{-1}|V^{I_{\mathfrak{p}}}\right)^{-1}, \tag{2.4}$$

where $\sigma_q \in \Gamma_K$ is a lift of the Frobenius element $\sigma_q \in \Gamma_{\mathbb{F}_K}$.

The following theorem interprets V^{I_K} as cohomology group of the inertial reduction. As a consequence L_p may be computed in characteristic p.

Theorem 2.8 *(a) There is a natural Γ_K-equivariant isomorphism*

$$V^{I_K} = H^1_{\mathrm{et}}(Y_{K^{\mathrm{alg}}}, \mathbb{Q}_\ell)^{I_K} \cong H^1_{\mathrm{et}}(\bar{Z}_k, \mathbb{Q}_\ell).$$

(b) The local L-factor $L_p(Y/K,s)$ equals the numerator of the local zeta function of $\bar{Z}$, i.e.

$$L_p(Y/K,s) = P_1(\bar{Z}, q^{-s})^{-1},$$

where

$$P_1(\bar{Z},T) := \det\left(1-\mathrm{Frob}_q \cdot T \mid H^1_{\mathrm{et}}(\bar{Z}_k, \mathbb{Q}_\ell)\right)$$

and $\mathrm{Frob}_q : \bar{Z} \to \bar{Z}$ is the relative q-Frobenius endomorphism with $q = |\mathbb{F}_K|$.

Proof: Part (a) is Theorem 2.4 of [3]. Part (b) follows from (a), see [3] Corollary 2.5. □

Remark 2.9 Assume that Y/K has good reduction. Then $L = K$ and we have that

$$L_{\mathfrak{p}}(Y/K,s) = P_1(q^{-s})^{-1},$$

where $P_1(T)$ is the numerator of the zeta function of the smooth curve $\bar{Y}/\mathbb{F}_K$ (a.k.a L-polynomial), which may be computed using point counting. Since $L = K$ the conductor exponent $f_{\mathfrak{p}}$ is trivial. (This follows immediately from (1.2) and the fact that the Swan conductor δ vanishes. We discuss this in more detail in § 2.3.)

Lemma 2.10 below gives a concrete description of the étale cohomology of the semistable curve $\bar{Z}_k$ together with the action of the absolute Galois group $\Gamma_{\mathbb{F}_K}$ on $\bar{Z}_k$. Together with Theorem 2.8.(b) this implies that we can compute the local L-factor $L_{\mathfrak{p}}(Y/K,s)$ from the explicit knowledge of the inertial reduction $\bar{Z}$. Before formulating the result, we need to introduce some more notation.

Denote by $\pi : \bar{Z}_k^{(0)} \to \bar{Z}_k$ the normalization. Then $\bar{Z}_k^{(0)}$ is the disjoint union of its irreducible components, which we denote by $(\bar{Z}_j)_{j\in J}$. These correspond to the irreducible components of $\bar{Z}_k$. The components $\bar{Z}_j$ are smooth projective curves. The absolute Galois group $\Gamma_{\mathbb{F}_K}$ of $\mathbb{F}_K$ naturally acts on the set of irreducible components. We denote the permutation character of this action by $\chi_{\rm comp}$.

Let $\xi \in \bar{Z}_k$ be a singular point. Then $\pi^{-1}(\xi) \subset \bar{Z}_k^{(0)}$ consists of two points. We define a 1-dimensional character ε_ξ on the stabilizer $\Gamma_{\mathbb{F}_K(\xi)} \subset \Gamma_{\mathbb{F}_K}$ of ξ as follows. If the two points in $\pi^{-1}(\xi)$ are permuted by $\Gamma_{\mathbb{F}_K(\xi)}$, then ε_ξ is the unique character of order two (*nonsplit case*). Otherwise, $\varepsilon_\xi = \mathbf{1}$ is the trivial character (*split case*). Denote by χ_ξ the character of the induced representation

$$\mathrm{Ind}_{\Gamma_{\mathbb{F}_K(\xi)}}^{\Gamma_{\mathbb{F}_K}} \varepsilon_\xi.$$

In the case that $\varepsilon_\xi = \mathbf{1}$ this is just the character of the permutation representation the absolute Galois group $\Gamma_{\mathbb{F}_K}$ acting on the orbit of ξ. Define

$$\chi_{\rm sing} = \sum_\xi \chi_\xi.$$

Here the sum runs over a system of representatives of the orbits of $\Gamma_{\mathbb{F}_K}$ acting on the set of the singularities of $\bar{Z}_k$ (these correspond exactly to the singularities of $\bar{Z}$).

We denote by $\Delta_{\bar{Z}_k}$ the graph of components of $\bar{Z}_k$.

The following lemma is Lemma 2.7 of [3].

Lemma 2.10 *Let $\bar{Z}/\mathbb{F}_K$ be a semistable curve and ℓ a prime with $\ell \nmid q$.*

(a) We have a decomposition

$$H^1_{\rm et}(\bar{Z}_k, \mathbb{Q}_\ell) = \oplus_{j\in J} H^1_{\rm et}(\bar{Z}_j, \mathbb{Q}_\ell) \oplus H^1(\Delta_{\bar{Z}_k})$$

as $\Gamma_{\mathbb{F}_K}$-representation.

(b) The character of $H^1(\Delta_{\bar{Z}_k})$ as $\Gamma_{\mathbb{F}_K}$-representation is $1 + \chi_{\rm sing} - \chi_{\rm comp} =: \chi_{\rm loops}$.

Lemma 2.10 implies that the local L-factor $L_{\mathfrak{p}}$ is the product of two factors: a contribution coming from the irreducible components and one from the loops in the graph $\Delta_{\bar{Z}_k}$ of components. For the polynomial P_1 we obtain

$$P_1(T) = P_{1,\rm comp}(T) \cdot P_{1,\rm loops}(T),$$

where

$$\begin{aligned} P_{1,\rm comp} &= \det\left(1 - \mathrm{Frob}_q \cdot T \mid H^1_{\rm et}(\bar{Z}_k, \mathbb{Q}_\ell)\right), \\ P_{1,\rm loops} &= \det\left(1 - \mathrm{Frob}_q \cdot T \mid H^1(\Delta_{\bar{Z}_k})\right). \end{aligned}$$

The polynomial $P_{1,\rm loops}$ can be easily computed from the description of the action of $\Gamma_{\mathbb{F}_K}$ on $H^1(\Delta_{\bar{Z}_k})$ given in Lemma 2.10.(b), see Example 3.11 for a concrete case. Note that

$$\begin{aligned} \deg(P_{1,\rm loops}) &= \chi_{\rm loops}(1) \\ &= 1 - \#\{\text{irreducible components}\} + \#\{\text{singularities}\} \end{aligned}$$

is the number of loops in the graph of components, and not twice this number as one might expect from the smooth case. Taking dimensions

We describe how to compute $P_{1,\mathrm{comp}}$. The irreducible components of $\bar{Z}$ are in general not absolutely irreducible. An irreducible component $\bar{Z}_{[j]}$ of $\bar{Z}$ decomposes in $\bar{Z}_k$ as a finite disjoint union of absolutely irreducible curves, which form an orbit under $\Gamma_{\mathbb{F}_K}$. Let $\bar{Z}_j$ be a representative of the orbit and let $\Gamma_j \subset \Gamma_{\mathbb{F}_K}$ be the stabilizer of $\bar{Z}_j$ and $\mathbb{F}_{q_j} = k^{\Gamma_j}$. The natural $\mathbb{F}_K$-structure of $\bar{Z}_{[j]}$ is given by

$$\bar{Z}_j/\Gamma_j \to \mathrm{Spec}(\mathbb{F}_{q_j}) \to \mathrm{Spec}(\mathbb{F}_K).$$

With this interpretation, the contribution of $\bar{Z}_{[j]}$ to the local zeta function in Theorem 2.8.(b) can be computed explicitly using point counting. We refer to § 5 of [3] for more details on how to compute the inertial reduction in the case of a superelliptic curve. An example where $\mathbb{F}_{q_j} \neq \mathbb{F}_K$ is discussed in § 7.2 of [3].

Example 2.11 This is an continuation of Example 2.3. We compute the local L-factor for $p = 11$ of

$$E: \; y^2 = f(x) = 4x^3 - 4x^2 + 1.$$

Recall that E has semistable reduction over $K = \mathbb{Q}_{11}$, and that $\bar{E}$ is a nodal cubic. Since $L = K$, the inertial reduction equals the special fiber of the quasi-stable model $\mathcal{E} = \mathcal{E}^{\mathrm{naive}}$.

The normalization $\bar{E}^{(0)}$ of $\bar{E}$ has genus zero, therefore

$$P_{1,\mathrm{comp}} = 1.$$

We compute the contribution of the loop. The curve $\bar{E}$ has an ordinary double point ξ in $\bar{x} = -3$. Recall that

$$\bar{f} = 4(\bar{x}+4)(\bar{x}+3)^2 \in \mathbb{F}_{11}[\bar{x}].$$

Since $4(-3+4) = 4 = 2^2$ is a square in $\mathbb{F}_{11}^{\times}$, we conclude that the two points of $\pi^{-1}(\xi)$ are $\Gamma_{\mathbb{F}_{11}}$-invariant, i.e. E has split multiplicative reduction. It follows that $\chi_{\mathrm{loops}} = \mathbf{1}$, i.e. $\Gamma_{\mathbb{F}_K}$ acts trivially on $H^1(\Delta_{\bar{Z}_k})$. Therefore $\varepsilon_\xi = 1$. We conclude that

$$L_{11}(\bar{Y}, T)^{-1} = P_{1,\mathrm{loops}} = (1 - \epsilon T) = (1 - T).$$

Since $\dim H^1_{\mathrm{et}}(\bar{Z}_k, \mathbb{Q}_\ell) = \dim H^1(\Delta_{\bar{Z}_k}) = 1$ and $L = K$, we conclude from (1.2) that

$$f_{11} = \epsilon_{11} = 2 - 1 = 1.$$

This is of course exactly what we expect for an elliptic curve with split multiplicative reduction.

2.3 The conductor exponent

In this section we give a formula for the conductor exponent $f_{Y/K}$ in terms of the reduction $\bar{Y}$ of a quasi-stable model $\mathcal{Y}$ of Y.

The conductor exponent is defined in (1.1) as $f_{Y/K} = \epsilon + \delta$. Theorem 2.8.(a) and (1.2) imply that

$$\epsilon = 2g(Y) - \dim H^1_{\mathrm{et}}(\bar{Z}_k, \mathbb{Q}_\ell). \tag{2.5}$$

Therefore ϵ may be computed from the inertial reduction $\bar{Z}$. The following observation follows from this, since the Swan conductor δ vanishes if L/K is at most tamely ramified.

Corollary 2.12 *Assume that L/K is at most tamely ramified. Then*

$$f_{Y/K} = 2g(Y) - \dim H^1_{\mathrm{et}}(\bar{Z}_k, \mathbb{Q}_\ell).$$

The following result expresses the Swan conductor δ in terms of the special fiber $\bar{Y}$ of a quasi-stable model $\mathcal{Y}$. Let $(\Gamma_i)_{i\geq 0}$ be the filtration of $\Gamma = \mathrm{Gal}(L/K)$ by higher ramification groups in the lower numbering. Recall that

$$\Gamma_i = \{\gamma \in \Gamma \mid v\left(\frac{\gamma(\pi_L) - \pi_L}{\pi_L}\right) \geq i\},$$

([13], Chapter 4). Writing the definition in this way takes ensures that we get the same numbers as in [13] even though we normalized the valuation differently. We therefore may write

$$I = \Gamma_0 \supsetneq \Gamma_1 = \cdots = \Gamma_{j_1} \supsetneq \Gamma_{j_1+1} = \cdots = \Gamma_{j_2} \supsetneq \cdots \Gamma_{j_r} \supsetneq \Gamma_{j_r+1} = \{1\}.$$

The breaks j_i in the filtration of higher ramification groups in the lower numbering are called the *lower jumps*.

The definition implies that $\Gamma_0 = I$ is the inertia group and $\Gamma_1 = P$ its Sylow p-subgroup. Let $\bar{Y}_i := \bar{Y}/\Gamma_i$ be the quotient curve. Then $\bar{Y}_0 = \bar{Y}/I = \bar{Z}_{\mathbb{F}_L}$ and $\bar{Y}_i = \bar{Y}$ for $i \gg 0$.

Theorem 2.13 *The Swan conductor is*

$$\delta = \sum_{i=1}^{\infty} \frac{|\Gamma_i|}{|\Gamma_0|} \cdot (2g(Y) - 2g(\bar{Y}_i)).$$

Here $g(\bar{Y}_i)$ denotes the arithmetic genus of $\bar{Y}_i$.

It is often more convenient to use a version of Theorem 2.13 in terms of the filtration of higher ramification groups in the upper numbering. Recall that the upper numbering is defined by

$$\Gamma^{\varphi(i)} = \Gamma_i,$$

where

$$\varphi(i) = \int_0^i \frac{\mathrm{d}t}{[\Gamma_0 : \Gamma_i]}$$

is the Herbrand function ([13], § 4.3). The breaks in the filtration of higher ramification groups in the upper numbering are called the *upper jumps*. We denote these breaks by $\sigma_1, \ldots, \sigma_r \in \mathbb{Q}$. The main advantage of the upper numbering is that they behave well under passing to a quotient ([13], Chapter 4, Prop. 14). This is convenient for computations, as it often allows one to work in a smaller field than L to compute the jumps. The following formula follows immediately from Theorem 2.13.

Corollary 2.14 *Write $\bar{Y}^u = \bar{Y}/\Gamma^u$. Then*

$$\delta = \int_0^\infty 2\left(g(Y) - g(\bar{Y}^u)\right)\,\mathrm{d}u.$$

3 Superelliptic curves: the tame case

Computing a quasi-stable model of a superelliptic curve in the case that the residue characteristic p does not divide the exponent of the superelliptic curve relies on the notion of admissible reduction. This approach is known and in principal also works for curves Y that admit a G-Galois cover $\phi : Y \to X = \mathbb{P}^1_K$ such that the residue characteristic p does not divide $|G|$. (We call this the *tame* case.) To compute the local L-factor and the conductor exponent we need to compute a Galois extension L/K, a quasi-stable $\mathcal{O}_K$-model $\mathcal{Y}$ together with the action of $\Gamma = \mathrm{Gal}(L/K)$ on $\mathcal{Y}$ explicitly.

3.1 Generalities on superelliptic curves

Definition 3.1 Let K be a field and n an integer which is prime to the characteristic of K. A *superelliptic curve of exponent* n defined over a field K is a smooth projective curve Y which is birationally given by an equation

$$Y :\ y^n = f(x), \tag{3.1}$$

where $f(x) \in K(x)$ is nonconstant.

For simplicity we always assume that $f \in \mathcal{O}_K[x]$.

A superelliptic curve of exponent n admits a map

$$\phi : Y \to X = \mathbb{P}^1_K, \qquad (x, y) \mapsto x.$$

The map ϕ is defined by x-coordinate, i.e. x is a coordinate on X and the function field of X is rational function field $K(x)$. If K contains a primitive nth root of unity ϕ is Galois, with Galois group $G = \mathbb{Z}/n\mathbb{Z}$.

Let K be a finite extension of $\mathbb{Q}_p$ and Y a superelliptic curve over K given by (3.1). Let L_0/K be the splitting field of f. We may write

$$f(x) = c \prod_{\alpha \in S} (x-\alpha)^{a_\alpha} \in L_0[x], \tag{3.2}$$

where $c \in K^\times$, $a_\alpha \in \mathbb{N}$ and the product runs over the set of roots S of f in L_0. We impose the following conditions on f and n.

Assumption 3.2 (a) We have and $\gcd(n, a_\alpha \mid \alpha \in S) = 1$.

(b) The exponent n is at least 2 and prime to p.

(c) We have $g(Y) \geq 1$.

Condition (a) ensures that Y is absolutely irreducible. The conditions $n \geq 2$ and $g(Y) \geq 1$ exclude some trivial cases. In particular, this condition ensures that ϕ branches at at least 3 points. The condition $p \nmid n$ ensures that we are in the tame case.

We note the affine curve $\mathrm{Spec}(L[x,y]/(y^n - f(x))$ is singular at the points with $x = \alpha$ if $a_\alpha \neq 1$. By computing the normalization of the local ring at such a point, we see that $\phi^{-1}(\alpha)$ consist of $\gcd(n, a_\alpha)$ points over the algebraic closure $\bar{K}$. Hence the ramification index of these points in ϕ is

$$e_\alpha = \frac{n}{\gcd(n, a_\alpha)}.$$

Similarly, the points of Y above $x = \infty$ have ramification index

$$e_\infty = \frac{n}{\gcd(n, \sum_\alpha a_\alpha)}.$$

From this one easily computes the genus of Y using the Riemann-Hurwitz formula. We denote by

$$D \subset X$$

the branch divisor of ϕ. The above calculation shows that $D_{L_0} \subset S \cup \{\infty\}$.

Note that the primes of bad reduction of f are contained in the finite set $\mathcal{S}$ of primes of K dividing $n \cdot c \cdot \Delta(\tilde{f})$, where $\tilde{f} = f/\gcd(f, f')$ is the radical of f and c is the leading coefficient of f. Namely, if $\mathfrak{p} \notin \mathcal{S}$ the naive model (Lemma 2.2) is smooth.

Now choose a finite extension L of K such that

Assumption 3.3

- L contains the splitting field L_0 of f,
- L contains an nth root of the uniformizing element π_{L_0} of L_0 and a primitive nth root of unity ζ_n,
- L/K is Galois.

Write $\Gamma = \mathrm{Gal}(L/K)$.

Theorem 3.4 *([3], Cor 4.6) Let Y/K be a superelliptic curve satisfying Assumption 3.2. Let L/K be a finite extension satisfying Assumption 3.3. Then the curve Y_L admits a quasi-stable model.*

The construction of the quasi-stable model $\mathcal{Y}$ of Y_L from Theorem 3.4 proceeds in two step.

(I) We construct the minimal semistable model $\mathcal{X}$ of X_L such that the branch points of ϕ specialize to pairwise distinct smooth points of the special fiber $\bar{X}$.

(II) The normalization of $\mathcal{X}$ in the function field of Y_L is quasi-stable.

The branch divisor D_L extends to a divisor $\mathcal{D} \subset \mathcal{X}$ which is étale over $\mathrm{Spec}(\mathcal{O}_L)$. The model $(\mathcal{X}, \mathcal{D})$ of (X_L, D_L) is called *stably marked* ([3], § 4.2). The special fiber $(\bar{X}, \bar{D})$ is stably marked in the sense that every irreducible component of $\bar{X}_k$ contains at least three points which are either singular or the specialization of a branch point of ϕ. The stably marked model exists and is unique since we assume that $|D_L| \geq 3$.

While normalizations are hard to compute in general, in our special situation it can be done explicitly.

3.2 Models of the projective line

In this section we compute the stably marked model $(\mathcal{X}, \mathcal{D})$ of (X_L, Y_L). More precisely, we describe the special fiber $(\bar{X}, \bar{D})$ of this model, together with the action of Γ. Since X has genus 0, the semistable curve $\bar{X}$ is a tree of projective lines. Each of the irreducible component $\bar{X}_v$ of $\bar{X}$ corresponds to the reduction of the projective line $\mathbb{P}^1_{\mathcal{O}_K}$ corresponding to some choice of coordinate x_v. Therefore a semistable curve $\mathcal{X}$ of genus 0 may be viewed as a 'set of coordinates'. We discuss this point of view in § 4.1.

We first introduce some notation. Let $\Delta = \Delta_{\bar{X}} = (V(\Delta), E(\Delta))$ denote the graph of components of $\bar{X}$. This is a finite, undirected tree whose vertices $v \in V(\Delta)$ correspond to the irreducible components $\bar{X}_v \subset \bar{X}$. Two vertices v_1, v_2 are adjacent if and only if the components $\bar{X}_{v_1}$ and $\bar{X}_{v_2}$ meet in a (necessarily unique) singular point of $\bar{X}$.

A *coordinate* on X_L is an L-linear isomorphism $x' : X_L \xrightarrow{\sim} \mathbb{P}^1_L$. Since we identify $X_L = \mathbb{P}^1_L$ via the chosen coordinate x, every coordinate may be represented by an element in $\mathrm{PGL}_2(L)$. We call two coordinates x_1, x_2 *equivalent* if the automorphism $x_2 \circ x_1^{-1} : \mathbb{P}^1_L \xrightarrow{\sim} \mathbb{P}^1_L$ extends to an automorphism of $\mathbb{P}^1_{\mathcal{O}_L}$, i.e. corresponds to an element of $\mathrm{PGL}_2(\mathcal{O}_L)$.

Let T denote the set of triples $t = (\alpha, \beta, \gamma)$ of pairwise distinct elements of D_L. For $t = (\alpha, \beta, \gamma)$ we let x_t denote the unique coordinate such that

$$x_t(\alpha) = 0, \quad x_t(\beta) = 1, \quad x_t(\gamma) = \infty.$$

Explicitly, we have

$$x_t = \frac{\beta - \gamma}{\beta - \alpha} \cdot \frac{x - \alpha}{x - \gamma}, \tag{3.3}$$

where we interpret this formula in the obvious way if $\infty \in \{\alpha, \beta, \gamma\}$. The equivalence relation $\sim$ induces an equivalence relation on T, which we denote by $\sim$ as well.

The following proposition is a reformulation of Lemma 5, together with the corollary to Lemma 4, of [6]. For more details we refer to [3], Proposition 4.2.

Proposition 3.5 *Let $(\mathcal{X}, \mathcal{D})$ be the stably marked model of (X_L, D_L). There is a bijection*

$$V(\Delta) \cong T/_{\sim}$$

between the set of irreducible components of $\bar{X}$ and the set of equivalence classes of charts.

For every $v \in V(\Delta)$ we choose a representative $t \in T$. The corresponding coordinate on X_L we denote by x_v. Equation (3.3) expresses x_v in terms of the original coordinate x of $X_L = \mathbb{P}^1_L$. We write $\bar{X}_v$ for the irreducible component of $\bar{X}$ corresponding to v. Reduction of the coordinate x_v induces an isomorphism $\bar{x}_v : \bar{X}_v \to \mathbb{P}^1_{\mathbb{F}_L}$.

Now let $\alpha \in D_L$ be a branch point. For every $v \in V(\Delta)$ the point $x = \alpha$ specializes to a unique point $\bar{x}_v(\alpha) \in \bar{X}_v$. This information determines the set $E(\Delta)$ of edges of the tree of components, which correspond to the singularities of $\bar{X}$. Moreover, the definition of T implies that there is a unique v such that the reduction $\bar{x}_v(\alpha) \in \bar{X}_v$ is not one of the singularities of $\bar{X}$. We illustrate this in a few concrete examples. More examples can be found in [3] §§ 5 and 6.

Example 3.6 We consider the elliptic curve Y from Example 2.4, which is defined by

$$Y : \; y^2 = f(x) = 4x^3 - 27.$$

We have seen that this elliptic curve has good reduction at all primes $p \neq 3$. The compute the model $\mathcal{X}$ for $p = 3$.

The polynomial $f(x)$ has roots

$$\alpha := 3 \cdot 2^{-2/3}, \quad \zeta_3\alpha, \quad \zeta_3^2\alpha,$$

where ζ_3 is a primitive root $3rd$ of unity. The splitting field of f is therefore $L_0 = \mathbb{Q}_3[2^{1/3}, \zeta_3]$ and $\mathrm{Gal}(L_0/K) = S_3$.

The coordinate x_1 corresponding to $t = (\alpha, -\zeta_3\alpha, \infty)$ equals

$$x = \alpha(\zeta_3 - 1)x_1 + \alpha, \tag{3.4}$$

by (3.3). For the specialization of the 4 branch points to the irreducible component $\bar{X}_1$ of $\bar{X}$ corresponding to the coordinate x_1 we find

$$\begin{array}{c|cccc} & \alpha & \zeta_3\alpha & \zeta_3^2\alpha & \infty \\ \hline \bar{x}_1 & 0 & 1 & -1 & \infty \end{array}.$$

To determine the specialization of $\zeta_3^2\alpha$ we note that

$$\frac{\zeta_3^2-1}{\zeta_3-1}=\zeta_3+1\equiv -1 \pmod{\pi_{L_0}}.$$

We conclude that the 4 branch points of ϕ specialize to 4 distinct points on $\bar{X}_1 \simeq \mathbb{P}^1_{\mathbb{F}_L}$. It follows that this is the unique irreducible component of $\mathcal{X}$, i.e. $\mathcal{X}\simeq\mathbb{P}^1_L$ with isomorphism given by x_1

The Galois group $\Gamma_0=\mathrm{Gal}(L_0/K)\simeq S_3$ acts on $\bar{X}_1$ by permuting the specializations of the three roots of f. This may also be deduced from the definition of x_1 by (3.4), by using that Γ_0 leaves the original coordinate x invariant.

For example let $\rho\in\Gamma_0$ be the element of order 3 with $\rho(\alpha)=\zeta_3\alpha$. Applying ρ to (3.4) we find for the (geometric) automorphism of $\bar{X}_1$ induced by ρ:

$$\begin{aligned} x &= \alpha(\zeta_3-1)x_1+\alpha \\ &= \zeta_3\alpha(\zeta_3-1)\rho(x_1)+\zeta_3\alpha. \end{aligned}$$

This yields $\rho(x_1)=\zeta_3^2x_1-\zeta_3^2$ which reduces to the "Artin–Schreier" automorphism

$$\bar{\rho}(\bar{x}_1)=\bar{x}_1-1$$

of $\bar{X}_1$. Similarly, one finds that the element $\sigma\in\Gamma_0$ which sends ζ_3 to ζ_3^2 induces

$$\sigma(x_1)=(\zeta_3^2+1)x_1-1.$$

Since L_0/K is a totally ramified extension, all elements of Γ_0 act L_0-linearly on X_{L_0}.

From the calculations we have done so far it is already clear that the elliptic curve Y has good reduction over a suitable extension L of L_0. The only thing that remains to be done is to compute the normalization $\mathcal{Y}$ of $\mathcal{X}$ in the function field of Y_L, for a field L satisfying Assumption 3.3.

3.3 Normalization

We keep the assumptions and notations of § 3.2. In particular, Y is a superelliptic curve defined over K given by (3.1) and L satisfies Assumption 3.3. We have already computed the stably marked model $(\mathcal{X},\mathcal{D})$ of (X_L,D_L) It remains it compute the normalization $\mathcal{Y}$ of $\mathcal{X}$ in the function field of Y_L. Cor. 4.6 of [3] shows that $\mathcal{Y}$ is a quasi-stable model of Y_L. Computing the model $\mathcal{Y}$ can be done piece by piece: it suffices to compute the restriction $\bar{Y}_v=\bar{Y}|_{\bar{X}_v}$ of $\bar{Y}$ to $\bar{X}_v$.

Let $v\in V(\Delta)$ and let x_v be the corresponding coordinate. We write η_v for the Gau valuation of the function field $L(x_v)$ of X_L, which extends the valuation of L. Recall that we have normalized the valuation on L by $v(p)=1$.

Notation 3.7 For every $v\in V(\Delta)$ we define

$$N_v=\frac{\eta_v(f)}{\eta_v(\pi_L)},\qquad f_v=\pi_L^{-N_v}f,\qquad y_v=\pi_L^{-N_v/n}y.$$

The definition of N_v ensures that $\eta_v(f_v)=0$ and we may consider the image $\bar{f}_v$ of f_v in the residue field $\mathbb{F}_L(\bar{x}_v)$ of the valuation η_v. The definition of y_v implies that we may cancel $\pi_L^{N_v/n}$ from the equation (3.1) of Y_L rewritten in terms of the new parameters x_v and y_v. We obtain

$$y_v^n = f_v(x_v). \tag{3.5}$$

The following statement one direction of [3], Proposition 4.5.(a).

Lemma 3.8 *The curve $\bar{Y}_v$ is semistable.*

The $\mathbb{F}_L$-curve $\bar{Y}_v$ is general not irreducible. The reason is that the restriction

$$\phi_v := \phi|_{\bar{X}_v} : \bar{Y}_v \to \bar{X}_v$$

has in general less branch points than ϕ, and the condition for absolute irreducibility analogous to Assumption 3.2.(a) need not be satisfied. (See Example 3.11.)

Example 3.9 We continue with Example 3.6. We choose $L = L_0[\sqrt[4]{-3}, i]$. Note that we have to adjoin the primitive 4th root of unity i to ensure that $L/\mathbb{Q}_3$ is Galois. For future reference we note that the inertia group $I \simeq C_3 \rtimes C_4$ is a binary dihedral group of order 12.

Rewriting f in terms of x_1 we find

$$f(x) = 4x^3 - 27 = 3^4\sqrt{-3}\left(x_1^3 + c_2x_1^2 + c_1x_1\right),$$

where $c_2 \equiv 0 \pmod{\pi_L}$ and $c_1 \equiv -1 \pmod{\pi_L}$. For an appropriate choice of the uniformizer π_L we have that $\pi_L^{N_v} = \left(\sqrt{-3}\right)^5$. Therefore we define

$$y = (\sqrt[4]{-3})^5 y_1.$$

In reduction we obtain the equation

$$\bar{Y}_1 : \; \bar{y}_1^2 = \bar{f}_1 = \bar{x}_1^3 - \bar{x}_1.$$

Let $\rho \in I$ be the element of order 3 which cyclically permutes the 3 roots of f as in Example 3.6. We have computed that ρ induces a nontrivial automorphism ρ on $\bar{X}_1$. The corresponding automorphism of $\bar{Y}_1$ is

$$\rho(\bar{X}_1, \bar{y}_1) = (\bar{x}_1 - 1, \bar{y}_1).$$

We conclude that the quotient curve $\bar{Y}_1/\langle\rho\rangle$, and hence $\bar{Z}_1 = \bar{Y}_1/\Gamma$, has genus zero, and hence does not contribute to the local L-factor. It follows that

$$L_{\mathfrak{p}} = 1.$$

It remains to compute the conductor exponent f_3. For this we compute the filtration of higher ramification groups of $I = \Gamma_0$. Since ρ generates the Sylow p-subgroup Γ_1 of I, it follows that $g(\bar{Y}_i) = 0$ for all i such that $\Gamma_i \neq \emptyset$. We leave it as an exercise to show that the filtration of higher ramification groups is

$$\Gamma_0 = I \supsetneq \Gamma_1 = \Gamma_2 = P = C_3 \supsetneq \Gamma_3 = \{1\}.$$

(One way to see this is to use the upper numbering for the higher ramification groups, and to note that $\mathrm{Gal}(L/L_0)$ is exactly the center of I.)

Equation (1.2) and Theorem 2.13 imply that

$$f_3 = \epsilon + \delta = 2 + 1 = 3.$$

This may also be computed by Ogg's formula.

Example 3.10 As a second example we consider the genus-2 curve Y over $\mathbb{Q}$ birationally given by

$$Y:\ y^2 = x^5 + x^3 + 3 =: f(x).$$

The discriminant of f is $\Delta(f) = 3^4 \cdot 3137$, therefore Y has good reduction to characteristic p for $p \neq 2, 3, 3137$. The wild case $p = 2$ we postpone until § 4.

We first determine the reduction at $p = 3$. Note that

$$f(x) \equiv x^3(x^2 + 1) \pmod{3}.$$

Therefore the special fiber $\bar{Y}^{\mathrm{naive}}$ of the naive model has one singularity $\xi = (0, 0)$. The normalization $\bar{Y}_0$ of $\bar{Y}^{\mathrm{naive}}$ has genus 1, and is defined by the affine equation

$$\bar{Y}_0:\ \left(\frac{\bar{y}}{\bar{x}}\right)^2 = \bar{x}(\bar{x}^2 + 1). \tag{3.6}$$

Over $\mathbb{Q}_3$ the polynomial f factors as $f = f_2 \cdot f_3$, where $f_2 \equiv x^2 + 1 \pmod{3}$ has degree 2 and f_3 is an Eisenstein polynomial of degree 3. In fact

$$f_3 \equiv x^3 + 6x^2 + 3 \pmod{9}.$$

We write $\mathbb{Q}_9$ for the unramified extension of $\mathbb{Q}_3$ of degree 2. The splitting field L_0 of f over $\mathbb{Q}_3$ is

$$L_0 = \mathbb{Q}_9[\alpha],$$

where α is a root of f_3. The extension $L_0/\mathbb{Q}_9$ is a totally ramified extension of degree 3. Since f_3 is an Eisenstein polynomial, α is a uniformizing element of L_0. We conclude that

$$L = L_0[\sqrt{\alpha}].$$

It follows that the inertia group I is cyclic of order 6.

Write $\alpha = \alpha_1, \alpha_2, \alpha_3 \in L$ for the three roots of f_3 in L. We define a new coordinate x_1 corresponding to $t = (\alpha, \alpha_2, \infty)$. The formula (3.3) yields

$$x = (\alpha_2 - \alpha)x_1 + \alpha.$$

We write $c := \alpha_2 - \alpha$. One computes that $v(c) = 2/3$. Rewriting f in terms of the variable x_1, one finds that

$$N_1 = \eta_1(f) = 2, \qquad f_1 = \frac{f}{c^3} \equiv \bar{x}_1^3 - \bar{x}_1 \pmod{\pi_L}, \qquad y = c^{3/2} y_1.$$

In reduction we find the following equation for the component $\bar{Y}_1$ of $\bar{Y}$:

$$\bar{Y}_1 : \; \bar{y}_1^2 = \bar{x}_1^3 - \bar{x}_1.$$

This is a smooth elliptic curve over the residue field $\mathbb{F}_L = \mathbb{F}_9$. The two components $\bar{Y}_0$ and $\bar{Y}_1$ intersect in a unique point in $\bar{Y}$: the unique singular point of $\bar{Y}$ is the point at ∞ of $\bar{Y}_1$, resp. the point with coordinates $\bar{x} = \bar{y} = 0$ on $\bar{Y}_0$.

The inertia group $I \simeq C_6$ acts nontrivially on $\bar{Y}_1$. Let $\rho \in I$ be the element of order 3 defined by

$$\rho(\alpha) = \alpha_2. \tag{3.7}$$

Then ρ induces the geometric automorphism

$$\rho(\bar{x}_1, \bar{y}_1) = (\bar{x}_1 + 1, \bar{y}_1).$$

The automorphism in I of order 2 induces the elliptic involution. (Actually to show this, it is not necessary to compute the precise coordinates x_1 and y_1.)

Therefore the quotient curve $\bar{Z}_1 := \bar{Y}_1/I$ has genus zero and does not contribute to the local L-factor. Since $\bar{Y}$ does not have loops, it follows that the local L-factor is determined by the zeta function of $\bar{Y}_0 = \bar{Z}_0$. Note that (3.6) is already the right model for $\bar{Y}_0$, since this is the normalization of the normal model. Point counting yields

$$L_3(\bar{Y}, T)^{-1} = 1 + 3T^2.$$

As a next step we compute the conductor exponent f_3. Since $I = C_6$, the filtration of higher ramification groups is

$$\Gamma_0 = I \supsetneq \Gamma_1 = \cdots = \Gamma_h = P = \langle \rho \rangle \supsetneq \Gamma_{h+1} = \{1\}.$$

We have computed that $g(\bar{Y}/I) = g(\bar{Y}/P) = 1$, therefore $g(\bar{Y}_i) = 1$ for $i = 0, \ldots, h$. Equation (2.5) yields that

$$\epsilon = 2 \cdot 2 - 2 \cdot 1 = 2.$$

It remains to compute the (unique) jump h in the filtration of higher ramification groups. We choose $\pi_L = \sqrt{\alpha}$ as uniformizing element of L. The definition of ρ in (3.7), together with the fact that $c = \rho(\alpha) - \alpha$ has valuation $2/3$, implies that

$$h = v\left(\frac{\rho(\pi_L) - \pi_L}{\pi_L}\right) = 2.$$

Theorem 2.13 implies that

$$\delta = 2\frac{3}{6}(2 \cdot 2 - 2 \cdot 1) = 2.$$

We conclude that
$$f_3 = \epsilon + \delta = 2 + 2 = 4.$$

We consider the local L-factor at $p = 3137$. We compute that
$$f(x) \equiv (x + 1556)^2 (x + 2366)(x^2 + 796x + 118) \pmod{p}.$$
As in Example 2.3 we conclude that $\bar{Y}^{\mathrm{naive}}$ is a semistable curve with one ordinary double point in $\bar{x} = -1556$, which is nonsplit.

Point counting on the elliptic curve $\bar{Y}_0$, defined as the normalization of $\bar{Y}^{\mathrm{naive}}$, yields
$$L_p^{-1} = (1 + pT^2)(1 + T).$$
Corollary 2.12 implies that
$$f_p = \epsilon = 2 \cdot 2 - 3 = 1,$$
since $H^1_{\mathrm{et}}(\bar{Z}_k, \mathbb{Q}_\ell) = H^1_{\mathrm{et}}(\bar{Y}_{0,k}, \mathbb{Q}_\ell) \oplus H^1(\Delta_{\bar{Y}_k})$ has dimension $2 \cdot 1 + 1 = 3$.

The following example illustrates some phenomena that can happen when the exponent of the superelliptic curve is not prime.

Example 3.11 As a further example we consider the genus-4 curve defined by
$$Y:\ y^4 = x(x-1)(x-3)^2(x-9) =: f(x).$$
Note that Y has good reduction for $p \neq 2, 3$. We compute the reduction at $p = 3$.

The curve $\bar{X}$ has three irreducible components. The following table lists the data from § 3.3. Here v is an index for the component and $t \in T$ is a corresponding triple of pairwise distinct branch points.

v	t	x_v	$\bar{f}_v$	N_v	y_v
0	$(0,1,\infty)$	$x_0 = x$	$\bar{f}_0 = \bar{x}^4(\bar{x}-1)$	0	$y_0 = y$
1	$(0,3,\infty)$	$x_1 = x/3$	$\bar{f}_1 = -\bar{x}_1^2(\bar{x}_1 - 1)^2$	4	$y_1 = y/3$
2	$(0,9,\infty)$	$x_2 = x/9$	$\bar{f}_2 = -\bar{x}_2(\bar{x}_2 - 1)$	6	$y_2 = y/3^{3/2}$.

The three irreducible components of $\bar{X}$ form a chain of projective lines: the singularity $\xi_1 = \bar{X}_0 \cap \bar{X}_1$ has coordinates $\bar{x} = 0, \bar{x}_1 = \infty$, the singularity $\xi_2 = \bar{X}_1 \cap \bar{X}_2$ has coordinates $\bar{x}_1 = 0, \bar{x}_2 = \infty$.

For the restriction $\bar{Y}_v = \bar{Y}|_{\bar{X}_v}$ we find:
$$\begin{aligned}
\bar{Y}_0 &: \bar{y}_0^4 = \bar{f}_0 = \bar{x}_0^4(\bar{x}_0 - 1),\\
\bar{Y}_1 &: \bar{y}_1^4 = \bar{f}_1 = -\bar{x}_1^2(\bar{x}_1 - 1)^2,\\
\bar{Y}_2 &: \bar{y}_2^4 = \bar{f}_2 = -\bar{x}_2(\bar{x}_2 - 1).
\end{aligned}$$

The explicit expressions for the coordinates y_i yields that we may take

$$L = \mathbb{Q}_3[\zeta_4, \sqrt{3}] = \mathbb{Q}_9[\sqrt{3}],$$

which is smaller than Assumption 3.3 suggests.

We write $\bar{\phi}_v : \bar{Y}_v \to \bar{X}_v$ for the map induced by ϕ. Note that $\bar{\phi}_0$ is unramified above $\bar{x}_0 = 0$, and $\bar{Y}_0$ has genus 0. The curve $\bar{Y}_1$ is reducible and consists of two projective lines, which we denote by $\bar{Y}_1^1$ and $\bar{Y}_1^2$. As $\mathbb{F}_L$-curves, these are birationally given by

$$\bar{Y}_1^j : \ \bar{y}_1^2 = (-1)^j \zeta_4 \bar{x}_1(\bar{x}_1 - 1). \tag{3.8}$$

The two curves $\bar{Y}_1^i$ are defined over $\mathbb{F}_9 = \mathbb{F}_3[\zeta_4]$, and are conjugate under the action of $\mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3)$. The curve $\bar{Y}_2$ is an elliptic curve. Since $\bar{\phi}_2 : \bar{Y}_2 \to \bar{X}_2$ is branched of order 2 at the point ξ_2, the inverse image $\phi^{-1}(\xi_2) \subset \bar{Y}_2$ consists of two points. Figure 3.1 illustrates the map $\bar{Y} \to \bar{X}$. The dots indicate the specializations of the branch points.

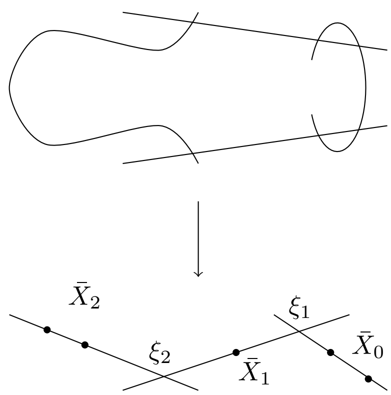


Figure 3.1: The map $\bar{Y} \to \bar{X}$

Let $\sigma \in I := \mathrm{Gal}(L/\mathbb{Q}_9)$ be the nontrivial automorphism. The ($\mathbb{F}_3$-linear) automorphism of $\bar{Y}$ induced by σ, acts on $\bar{Y}_2$ as

$$\sigma(\bar{x}_2, \bar{y}_2) = (\bar{x}_2, -\bar{y}_2).$$

Since $\sigma \in \mathrm{Gal}(\bar{Y}_2/\bar{X}_2) \simeq C_4$, we find for $\bar{Z}_2 = \bar{Y}_2/I = \bar{Y}_2/\Gamma$

$$\bar{Z}_2 : \ \bar{z}_2^2 = -\bar{x}_2(\bar{x}_2 - 1), \qquad \bar{z}_2 := \bar{y}_2^2.$$

In particular, it follows that σ fixes the two points of $\bar{Y}_2$ above ξ_2. Hence σ also leaves the two irreducible components $\bar{Y}_1^1$ and $\bar{Y}_2^2$ invariant. This also directly follows from (3.8). The automorphism σ acts trivially on $\bar{Y}_0$. We conclude that $\bar{Z}_k$ consists of 4 projective lines intersecting in 6 ordinary double points.

Since all irreducible components of $\bar{Z}_k$ have genus 0, he local L-factor only has a contribution coming from the cohomology of the graph of components of $\bar{Z}_k$. Note that $\bar{Z}_k$ has arithmetic genus 3.

We compute the polynomial $P_{1,\text{loops}}$.

We have already seen that the 4 irreducible components of $\bar{Z}_k$ form 3 orbits under the action of $\Gamma_{\mathbb{F}_3}$. Note that the 6 singularities form 3 orbits under the action of $\Gamma_{\mathbb{F}_3}$, which are each of length 2. By considering the action of $\Gamma_{\mathbb{F}_3}$ on the irreducible components one sees that all singularities are split, i.e. $\varepsilon = 1$ for all singularities of $\bar{Z}_k$. This implies that the character χ_{sing} is just the character of the permutation representation of $\mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3)$ acting on the singularities. An elementary calculation shows that the character χ_{loops} from Lemma 2.10.(b) is the character of $\mathbf{1} + 2 \cdot (\mathbf{-1})$, where $\mathbf{-1}$ is the nontrivial character of $\mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3) \simeq C_2$. We conclude that

$$L_3^{-1} = P_{1,\text{loops}}(T) = (1-T)(1+T)^2.$$

Since $L(\mathbb{Q}_3)$ is tame, Corollary 2.12 implies that

$$f_3 = \epsilon = 2g(Y) - \dim H^1_{\text{et}}(\bar{Z}_k, \mathbb{Q}_\ell) = 2 \cdot 4 - 3 = 5.$$

3.4 Interpretation in terms of admissible covers

In the situation of this section the covers $\bar{\phi} : \bar{Y} \to \bar{X}$ we constructed are so-called admissible covers. Admissible covers arise naturally as degenerations of covers between smooth curves that are at most tamely ramified.

We fix a compatible system of roots of unity $(\zeta_n)_n$ of order prime to the characteristic of the residue field. Let $\phi : Y \to X$ be a tame G-Galois cover between semistable curves. Let $y \in Y^{\text{sm}}$ be a smooth point which is ramified in ϕ. The *canonical generator of inertia* is the element g of the stabilizer G_y of y such that

$$g^* u \equiv \zeta_{|G_y|} u \pmod{u^2},$$

where $u = u_y$ is a local parameter at y. Note that the canonical generator of inertia depends on the choice of the compatible system of roots of unity.

Definition 3.12 Let $\bar{\phi} : \bar{Y} \to \bar{X}$ be a tame G-Galois cover between semistable curves defined over an algebraically closed field. The cover $\bar{\phi}$ is called *admissible* if the following conditions are satisfied.

(a) The singular points of $\bar{Y}$ map to singular points of $\bar{X}$.

(b) For each singular point $\xi \in \bar{Y}$ the canonical generators of inertia corresponding to the two branches of $\bar{Y}$ at y are inverse to each other.

The following results is a version in our context of a more general result.

Proposition 3.13 *Let $\phi : Y \to X = \mathbb{P}^1_L$ be the n-cyclic cover associated with a superelliptic curve Y of index n over a field L satisfying Assumption 3.3. Let $\mathcal{X}$ be the stably marked model of X, and $\mathcal{Y}$ the normalization of $\mathcal{X}$ in the function field of Y. Then the reduction $\bar{\phi} : \bar{Y} \to \bar{X}$ is an admissible cover.*

Let Y be a superelliptic curve of exponent n and $\phi : Y_L \to X_L = \mathbb{P}^1_L$ the associated Galois cover with Galois group $G \simeq C_n$. We let $g \in G$ be the generator with

$$g(x, y) = (x, \zeta_n),$$

where ζ_n is the fixed primitive nth root of unity. Let $\alpha \in \mathbb{P}^1_L \setminus \{\infty\}$ be a branch point. Since ϕ is Galois, the canonical generator of inertia of α does not depend on the choice of a point in $\phi^{-1}(\alpha)$. Define

$$a_\alpha = \mathrm{ord}_{x=\alpha}(f).$$

Then the canonical generator of inertia of α is g^{a_α}.

From this information it is easy to deduce the cover $\bar{\phi} : \bar{Y} \to \bar{X}$ from the knowledge of $\bar{X}$. We leave it as an exercise to work this out in Example 3.11.

The definition of $\mathcal{Y}$ as the normalization of $\mathcal{X}$ in the function field of Y implies that the action of Γ on $\bar{Y}$ commutes with the map $\bar{\phi} : \bar{Y} \to \bar{X}$, and therefore induces an action on $\bar{X}$. The action of Γ on $\bar{X}$ is completely determined by the action of Γ on the branch points. The information of the action of Γ on $\bar{X}$ already contains a large part of the information needed. For example, from this information it may often be seen that an irreducible component $\bar{Z}_v$ has genus zero and hence does not contribute to the local L-factor. In these cases it is not necessary to compute the precise model of $\bar{Z}_v$.

4 Superelliptic curves: the wild case

As before, let $K/\mathbb{Q}_p$ be a finite extension and Y/K a superelliptic curve given generically by the equation

$$y^n = f(x),$$

where $f \in K[x]$ is a polynomial. It is no restriction to assume that the exponent of any irreducible factor of f is strictly smaller than n. In the previous section we have assumed that n is prime to p (the tame case). In this section we assume that $n = p$, and we explain a method for computing the stable reduction of Y (building on work of Coleman ([4]) and Matignon and Lehr ([10], [8])). More details and complete proofs can be found in [12].

Combining the methods of Section 3 and this section, it should be possible to obtain an algorithm to compute the semistable reduction of superelliptic curves for all exponents n. We plan to work out special cases (e.g. $n = p^2$) in the future. It is however clear that things get very complicated if the exponent of p in n gets large.

As before we consider the given superelliptic curve Y as a cover of the projective line $X = \mathbb{P}^1_K$. Here the cover map $\phi : Y \to X$ corresponds to the extension of function fields $K(Y)/K(X)$ generated by an element y with minimal equation $y^p = f(x)$. As in Section 3, our general strategy is to find a finite extension L/K and a semistable model $\mathcal{X}$ of X_L with the property that its normalization $\mathcal{Y}$ with respect to ϕ is again semistable. But this is more or less the only similarity to the tame case. It is typically much more difficult to find the right extension L/K and the semistable model $\mathcal{X}$ of X_L.

4.1 Two easy examples

We start with two easy but illustrative examples. In both examples, $p = 2$, $K = \mathbb{Q}_2$ and the polynomial f has degree 3 and three distinct zeroes. Hence our curve Y is an elliptic curve over $\mathbb{Q}_2$ in (almost) Weierstrass normal form,

$$Y:\ y^2 = f(x).$$

(We do not assume that f is monic.)

Example 4.1 We consider the curve

$$Y:\ y^2 = f(x) := 4 + x^2 + 4x^3 \tag{4.1}$$

over $\mathbb{Q}_2$.

Let us first compute the minimal model of Y. Using `Sage`, we obtain the minimal Weierstrass equation

$$w^2 + xw = x^3 + 1. \tag{4.2}$$

The above equation defines an elliptic curve over $\mathbb{Z}_2$ (even over $\mathbb{Z}[433^{-1}]$). In particular, Y has good reduction at $p = 2$.

How does the transformation from (4.1) to (4.2) work in general? The trick is to find polynomials $g, h \in \mathbb{Z}_2[x]$ such that

$$f = h^2 + 4g. \tag{4.3}$$

In our example, $h = x$ and $g = 1 + x^3$. If we substitute $y = h + 2w$ into the equation $y^2 = f(x)$ we obtain, after a short calculation (do it yourself!) the equation

$$w^2 + h(x)w = g(x). \tag{4.4}$$

This equation still defines the same plane affine curve over $\mathbb{Q}_2$, because we can also write $w = (y - h)/2$. However, both equations define different plane affine curves over $\mathbb{Z}_2$. In fact, (4.4) defines a finite cover of the model defined by (4.3). If Equation (4.4) reduces to an irreducible equation over $\mathbb{F}_L$, then it defines in fact a normal model and thus the normalization of the model defined by (4.3). Compare with Lemma 2.2.

In more general examples (for instance, for polynomials f of degree > 4) this may get considerably more complicated. Nevertheless, variations of the same trick (i.e. writing f in the form (4.3) and then substituting $y = h + 2w$) will turn out to be very useful in general.

Example 4.2 We consider the curve

$$Y:\ y^2 = f(x) := 1 + 2x + x^3 \tag{4.5}$$

over $\mathbb{Q}_2$. Using again `Sage` we check that the given equation is in fact a minimal Weierstrass equation which defines a regular model $\mathcal{Y}_0$ of Y over $\mathbb{Z}_2$. Nevertheless, $\mathcal{Y}_{0,s}$ is singular, and hence Y has bad reduction.

Note that the model $\mathcal{Y}_0$ is the normalization of the smooth model $\mathcal{X}_0 := \mathbb{P}^1_{\mathbb{Z}_2}$ of $X = \mathbb{P}^1_K$ (where $K = \mathbb{Q}_2$). In other words, $\mathcal{Y}_0 = \mathcal{Y}^{\mathrm{naive}}$ is the naive model as defined in § 2.1. In particular, the cover $\phi : Y \to X$ extends to a finite map $\mathcal{Y}_0 \to \mathcal{X}_0$. Its restriction $\mathcal{Y}_{0,s} \to \mathcal{X}_{0,s}$ to the special fiber looks as in Figure 4.1.

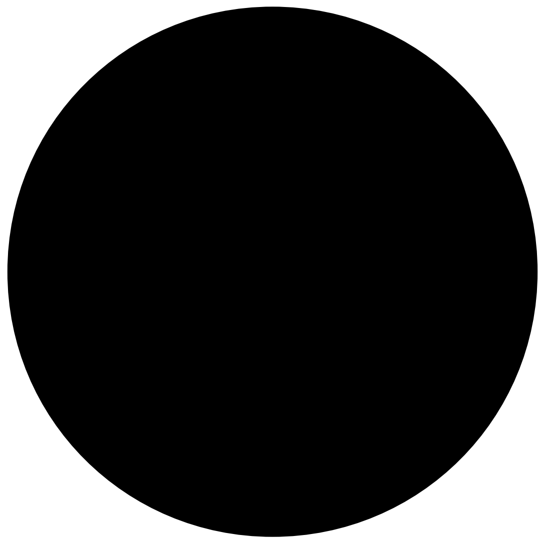


Figure 4.1: The map $\mathcal{Y}_{0,s} \to \mathcal{X}_{0,s}$

To make sense of Figure 4.1, first note first that $\mathcal{X}_{0,s} = \mathbb{P}^1_{\mathbb{F}_2}$ is the projective line over $\mathbb{F}_2$ (with coordinate x). The four dashes on $\mathcal{X}_{0,s}$ represent the specializations of the four branch point of the cover $\phi : Y \to X = \mathbb{P}^1_K$ (which are ∞ and the three zeros of f). The fact that all four branch points specialize to distinct points on $\mathcal{X}_{0,s}$ corresponds to the fact that the image of f in $\mathbb{F}_2[x]$,

$$\bar{f} = x^3 + 1 = (x+1)(x^2 + x + 1) \in \mathbb{F}_2[x],$$

is separable. The affine part of $\mathcal{Y}_{0,s}$ (the inverse image of $\operatorname{Spec} \mathbb{F}_2[x] \subset \mathcal{X}_{0,s}$) is given by the equation

$$y^2 = \bar{f}(x) = x^3 + 1.$$

Therefore, the map $\mathcal{Y}_{0,s} \to \mathcal{X}_{0,s}$ is a finite, flat and purely inseparable homeomorphism of degree 2. Using the Jacobian criterion we see that the singular points on the affine part of $\mathcal{Y}_{0,s}$ are precisely the points which lie over the zeroes of $\bar{f}'$. Since

$$\bar{f}' = (x^3 + 1)' = x^2,$$

$\mathcal{Y}_{0,s}$ has a unique singularity above the point $x = 0$ on $\bar{X}_{0,s}$. After substituting $y = 1 + w$ we obtain the equation

$$w^2 = x^3$$

which shows that the singularity is a cubic cusp.

So it seems that Y has bad reduction. On the other hand, the j-invariant of Y, $j(Y) = 55296/59$, is 2-integral. This shows that Y has potentially good reduction: there exists a finite extension L/K and a smooth model $\mathcal{Y}$ of Y_L. The elliptic involution ι (which is $y \mapsto -y$ on the generic fiber Y_L) extends to $\mathcal{Y}$. Its restriction to the special fiber $\mathcal{Y}_s$ is again a nontrivial involution; after all $\mathcal{Y}_s$ is an elliptic curve. Let $\mathcal{X} := \mathcal{Y}/\langle\iota\rangle$ denote the quotient. Then $\mathcal{X}$ is a normal model of $X = \mathbb{P}^1_L$ and $\mathcal{Y} \to \mathcal{X}$ is a finite morphism. Furthermore, because the restriction of ι to $\mathcal{Y}_s$ is nontrivial, its restriction to the special fiber $\bar{\phi} : \mathcal{Y}_s \to \mathcal{X}_s = \mathcal{Y}_s/\langle\iota\rangle$ is the quotient map of ι and hence a finite, generically étale morphism between smooth projective curves. In particular, $\mathcal{X}$ is a smooth model of $X = \mathbb{P}^1_L$. This means that $\mathcal{X} \cong \mathbb{P}^1_{\mathcal{O}_L}$, and the isomorphism corresponds to a new coordinate x_1 which is given as a fractional linear transformation, with coefficients in L, of our original coordinate x (§ 3.2).

Claim: the coordinate transformation between x and x_1 can be assumed to be of the form

$$x = \alpha + \beta x_1,$$

where $\alpha, \beta \in \mathfrak{m}_L$ are elements of the maximal ideal of the valuation ring of L; in other words, $v_L(\alpha), v_L(\beta) > 0$.

To prove this claim, we consider the semistable model $\mathcal{X}_1$ of X_L corresponding to the two coordinates x and x_1 simultaneously. This is similar to the correspondence in Proposition 3.5. In other words, $\mathcal{X}_1$ is the minimal model of X_L which dominates both $\mathcal{X}_0$ and $\mathcal{X}$. Its special fiber $\mathcal{X}_{1,s}$ consists of two smooth components $\bar{X}_0, \bar{X}_1$ intersecting in a unique ordinary double point. We have natural isomorphisms $\bar{X}_i \cong \mathbb{P}^1_{\mathbb{F}_l}$ corresponding to the restrictions of the coordinate functions x and x_1 to $\mathcal{X}_{1,s}$. The above claim is equivalent to the statement that the point where the two components intersect is given by $x = 0$ on $\bar{X}_0$. Phrased like this, the claim looks very reasonable in view of Figure 4.1: it seems obvious that we have to modify the model $\mathcal{X}_0$ at the point on $\mathcal{X}_{0,s}$ given by $x = 0$.

To make this argument more rigorous, let $\mathcal{Y}_1$ be the normalization of $\mathcal{X}_1$ with respect to ϕ. The maps between the various models are represented in the following diagram:

$$\begin{tikzcd}
 & \mathcal{Y}_1 \arrow[dl] \arrow[dd] \arrow[dr] & \\
\mathcal{Y}_0 \arrow[ddd] & & \mathcal{Y} \arrow[ddd] \\
 & \mathcal{X}_1 \arrow[ddl] \arrow[ddr] & \\
 & & \\
\mathcal{X}_0 & & \mathcal{X}
\end{tikzcd}$$

We can infer from our knowledge about the maps $\mathcal{Y}_0 \to \mathcal{X}_0$ and $\mathcal{Y} \to \mathcal{X}$ that the induced map $\mathcal{Y}_{1,s} \to \mathcal{X}_{1,s}$ looks as in Figure 4.2.

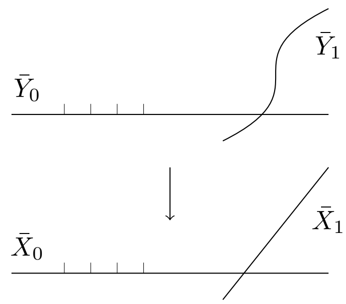


Figure 4.2: The map $\mathcal{Y}_{1,s} \to \mathcal{X}_{1,s}$.

The special fiber $\mathcal{Y}_{1,s}$ of $\mathcal{Y}_1$ consists of two components $\bar{Y}_0$ and $\bar{Y}_1$ which meet in a single point. The restriction of the map $\mathcal{Y}_{1,s} \to \mathcal{X}_{1,s}$ to the affine line $\bar{Y}_0\backslash\bar{Y}_1$ can be identified with the restriction of the map $\mathcal{Y}_{0,s} \to \mathcal{X}_{0,s}$ to the complement of some closed point. Similarly, the restriction to the affine line $\bar{Y}_1\backslash\bar{Y}_0$ can be identified with the restriction of the map $\mathcal{Y}_{1,s} \to \mathcal{X}_{1,s}$ to the affine part of the elliptic curve $\mathcal{Y}_{1,s}$. In particular, the component $\bar{Y}_1$ is an elliptic curve, while $\bar{Y}_0$ is a projective line.

The map $\mathcal{Y}_1 \to \mathcal{Y}_0$ contracts the component $\bar{Y}_1$ and is an isomorphism everywhere else. Since $\mathcal{Y}_{1,s}$ is smooth on the complement of $\bar{Y}_1$, this implies that the image of $\bar{Y}_1$ must be the unique singular point of $\mathcal{Y}_{0,s}$. This completes the proof of the claim.

Remark 4.3 The above argument becomes more intuitive if we use language from rigid analytic geometry. Let X^{an} and Y^{an} denote the rigid-analytic spaces associated to the curves X and Y (in the sense of Tate, or of Berkovic if you prefer). To the choice of a model corresponds a *specialization map* from the generic to the special fiber of the model. The inverse image of an open or closed subset of the special fiber then defines an admissible open subset of the generic fiber. For instance, consider the specialization map

$$\mathrm{sp}_{\mathcal{X}_0} : X^{\mathrm{an}} \to \mathcal{X}_{0,s} \cong \mathbb{P}^1_{\mathbb{F}_2}.$$

The inverse image of the point $x = 0$ consists of all points $x = \alpha$ with $v_K(\alpha) > 0$ and can therefore be identified with the *open unit disk*,

$$D_0^\circ := \{x \mid v_K(x) > 0\} \subset X^{\mathrm{an}}.$$

If we use the model $\mathcal{X}_1$ instead, then D_0° may also be represented as the inverse image of the component $\bar{X}_1$, because $\bar{X}_1$ is contracted to the point $x = 0$ under the map $\mathcal{X}_1 \to \mathcal{X}_0$. Set

$$D := \mathrm{sp}_{\mathcal{X}_1}^{-1}(\bar{X}_1 \backslash \bar{X}_0) \subset X_L^{\mathrm{an}}.$$

Then $D \subset (D_0^\circ)_L$ is an affinoid subdomain. Moreover, since $\bar{X}_1 \backslash \bar{X}_0$ is isomorphic to the affine line over $\mathbb{F}_L$, D is isomorphic to a *closed (affinoid) disk*. This means that there exists $\alpha, \beta \in \mathcal{O}_L$ with $v_L(\alpha), v_L(\beta) > 0$ such that

$$D = \{x \mid v_L(x - \alpha) \geq v_L(\beta)\}.$$

Note that the condition defining D is equivalent to $v_L(x_1) \geq 0$, where $x_1 = (x - \alpha)/\beta$ is the new coordinate we are looking for.

As a very special case of a very general theory (see [2]) we obtain a bijective correspondence between

- closed affinoid disks $D \subset (D_0^\circ)_L$, and
- modifications $\mathcal{X}_1 \to \mathcal{X}_{0,L} := \mathbb{P}^1_{\mathcal{O}_L}$ which are an isomorphism outside the point $x = 0$ on $\mathcal{X}_{0,s}$ and whose exceptional divisor is a projective line.

The main question we face is: how do we find the critical closed disk $D \subset D_0^\circ$ (or, equivalently, the transformation $x = \alpha + \beta x_1$)? It is natural to first try to find the correct center α of D and then determine the radius $r = v_L(\beta)$. Therefore, we write $x = \alpha + t$, $t = \beta x_1$ and f as a polynomial in t and x_1:

$$\begin{aligned} f &= 1 + 2x + x^3 = 1 + 2(\alpha + t) + (\alpha + t)^3 \\ &= a_0 + a_1 t + a_2 t^2 + a_3 t^3 \\ &= a_0 + a_1 \beta x_1 + a_2 \beta^2 x_1^2 + a_3 \beta^3 x_1^3, \end{aligned}$$

where

$$\begin{aligned} a_0 &= f(\alpha) = 1 + 2\alpha + \alpha^3, \\ a_1 &= f'(\alpha) = 2 + 3\alpha^2, \\ a_2 &= \frac{1}{2} f''(\alpha) = 3\alpha, \\ a_3 &= \frac{1}{6} f'''(\alpha) = 1. \end{aligned} \tag{4.6}$$

Let us fix for the moment some Galois extension L/K and a transformation $x = \alpha + \beta x_1$ with $\alpha, \beta \in \mathcal{O}_L$, and let $\mathcal{X} \cong \mathbb{P}^1_{\mathcal{O}_L}$ be the smooth model of X_L corresponding to the coordinate x_1. Let us assume that there exists a decomposition of f of the form

$$f = h^2 + 4g, \qquad h, g \in \mathcal{O}_L[x_1]. \tag{4.7}$$

Substituting $y = h + 2w$ into the equation $y^2 = f$ and using (4.7) we obtain the new equation

$$w^2 + h(x_1)w = g(x_1). \tag{4.8}$$

Assuming that the reduction of this equation to $\mathbb{F}_L$ defines a reduced affine curve, we can use the argument from Lemma 2.2 to show that the normalization $\mathcal{Y}$ of $\mathcal{X}$ with respect to ϕ is given by (4.8). Assuming, moreover, that (4.8) reduces to an equation of an elliptic curve over $\mathbb{F}_L$, we can conclude that $\mathcal{Y}$ is in fact the (unique) smooth model of Y_L. This means that our choice of L/K and $\alpha, \beta \in \mathcal{O}_L$ was good.

But how do we find the extension L/K and the right transformation $x = \alpha + \beta x_1$? Let us first try the most naive guess, i.e. set $L := K = \mathbb{Q}_p$ and $\alpha = 0$. We substitute $x = \beta x_1$ into f,

$$f = 1 + 2\beta x_1 + \beta^3 x_1^3.$$

It is easy to see that a decomposition as in (4.7) exists iff $v_K(\beta) \geq 1$. Hence we try $\beta := 2$, i.e. $x = 2x_1$. This corresponds to choosing for D the closed disk $D(0,1)$ with center $\alpha = 0$ and radius $v_K(\beta) = 1$. Then

$$f = 1 + 4x_1 + 8x_1^8 = 1^2 + 4(x_1 + 2x_1^3).$$

Substituting $y = 1 + 2w$ we obtain the equation

$$w^2 + w = x_1 + 2x_1^3.$$

The special fiber of the corresponding model has an affine open subset given by the equation

$$w^2 + w = x.$$

Unfortunately, this is a curve of genus 0 and therefore not what we were looking for. Actually, we really need to obtain the equation $w^2 + w = x^3$. The trouble is caused by the coefficient of X_1 in the polynomial

$$g = x_1 + 2x_1^3$$

which has lower valuation than the coefficient of x_1^3. It is obviously no use to vary the radius of the disk (i.e. choose another β). No disk with center $\alpha = 0$ will work.

The trick to find the right center is to first write the decomposition of f with generic coefficients and such that the coefficient of $t = \beta x_1$ *and* of $t^2 = \beta^2 x_1^2$ in g vanishes:

$$\begin{aligned} f = a_0 + a_1 t + a_2 t^2 + a_3 t^3 &= h^2 + 4g \\ &= (b_0 + b_1 t)^2 + 4(c_2 t^2 + c_3 t^3). \end{aligned} \tag{4.9}$$

Comparing coefficients, we see that the equality in (4.9) is equivalent to the system of equations

$$\begin{aligned} a_0 &= b_0^2, \\ a_1 &= 2b_0 b_1, \\ a_2 &= b_1^2 + 4c_2, \\ a_3 &= 4a_3. \end{aligned} \tag{4.10}$$

Solving these equations for b_i, c_k we obtain

$$\begin{aligned} b_0 &= a_0^{1/2}, \\ b_1 &= \frac{a_1}{2a_0^{1/2}}, \\ c_2 &= \frac{a_0a_2 - a_1^2}{8a_0}, \\ c_3 &= \frac{a_3}{4}. \end{aligned} \tag{4.11}$$

The most interesting coefficient is c_2, which can be expressed in terms of the center α as

$$c_2 = \frac{d_2(\alpha)}{8f(\alpha)},$$

with

$$d_2 := \frac{1}{2}ff'' - (f')^2 = \frac{3}{4}(x^4 + 4x^2 + 4x - 4/3) \in K[x]. \tag{4.12}$$

Let L_0/K be the splitting field of the (irreducible!) polynomial $d_2 \in K[x]$ and let $\alpha \in L_0$ be a root of d_2. Let L/L_0 be the quadratic extension obtained by adjoining the square root

$$\delta := f(\alpha)^{1/2} = (1 + 2\alpha + \alpha^3)^{1/2}.$$

After substituting $x = \alpha + t$ we can write

$$f = h^2 + 4g,$$

with

$$h = b_0 + b_1 t = \delta + \frac{2 + 3\alpha^2}{2\delta}t, \qquad g = \frac{1}{4}t^3.$$

We see that we are on the right track. We still need the substitution $t = \beta x_1$, and all that matters is that $v_L(\beta) = 2/3$. One checks that $\beta := \alpha - \alpha_1$ works, where α_1 is another root of d_2 distinct from α. Substituting $x = \alpha + \beta x_1$ and $y = h + 2w$ we obtain the equation

$$w^2 + hw = \frac{\beta^3}{4}x^3$$

with coefficients in $\mathcal{O}_L$. Reduction modulo the maximal ideal of $\mathcal{O}_L$ yields the equation

$$w^2 + w = x^3 \tag{4.13}$$

over $\mathbb{F}_L$ (we have also used $h \equiv \delta \equiv 1$ and $\beta^3/4 \equiv 1$). Obviously, (4.13) is the equation of a supersingular elliptic curve over $\mathbb{F}_L$. It follows that Y_L has good reduction, and that its reduction $\bar{Y}$ is the curve (4.13).

Let us now analyze the action of $\Gamma = \mathrm{Gal}(L/K)$ on the reduction $\bar{Y}$. We have seen in§ 3.4 that this action commutes with the cover $\bar{\phi} : \bar{Y} \to \bar{X} \cong \mathbb{P}^1_{\mathbb{F}_L}$

and therefore induces an action on $\bar{X}$. We claim that this action factors through to a faithful action of the quotient group $\Gamma_0 = \mathrm{Gal}(L_0/K)$,

$$\Gamma_0 \hookrightarrow \mathrm{Aut}(\bar{X}).$$

To see this, recall that L_0 was defined to be the splitting field of the quartic polynomial d_2, and that the isomorphism $\mathcal{X} \cong \mathbb{P}^1_{\mathcal{O}_L}$ corresponds to the choice of the coordinate

$$x_0 = \frac{x-\alpha}{\beta},$$

where α is a zero of d_2. Therefore, the claim follows from the assertion that the four zeroes of d_2 specialize to four distinct points on $\bar{X} \cong \mathbb{P}^1_{\mathbb{F}_l}$. The last claim holds because β was defined as the difference between two distinct zeroes of d_2. In the language of rigid geometry, this means that the closed disk

$$D := \{x_1 \mid v_L(x_1) \geq 0\} = \{x \mid v_L(x-\alpha) \geq 2/3\}$$

is the smallest disk containing all four zeroes of d_2.

Using `Sage` and the *Database of Local Fields* ([7]), we check the following:

- The permutation representation on the roots on d_2 defines an isomorphism $\Gamma_0 \cong S_4$. The inertia subgroup $I_0 \subset \Gamma_0$ corresponds to A_4.
- The Galois group $\Gamma = \mathrm{Gal}(L/K)$ is isomorphic to the group $\mathrm{GL}_2(3)$ of order 48, with inertia subgroup $I \subset \Gamma$ corresponding to $\mathrm{SL}_2(3)$. In particular, the extension L/L_0 is wildly ramified of degree 2.

Let σ be the unique nontrivial element of $\mathrm{Gal}(L/L_0)$. By definition of L we have $\sigma(\delta) = -\delta$. As in Example 3.6 we compute the monodromy action of σ on $\bar{Y}$ via its effect on the function $w = (y-h)/2$. We obtain

$$\begin{aligned} \sigma(w) &= \frac{y-\sigma(h)}{2} = w + \frac{h-\sigma(h)}{2} \\ &\equiv w + \frac{\delta - \sigma(\delta)}{2} \pmod{\mathfrak{m}_L} \\ &\equiv w + 1. \end{aligned}$$

This shows that σ acts on $\bar{Y}$ as the elliptic involution. In particular, the action of Γ on $\bar{Y}$, as well as its restriction to the inertia group I, is faithful. Note that we obtain an isomorphism

$$I \cong \mathrm{Aut}_{\bar{\mathbb{F}}_2}(\bar{Y}_{\bar{\mathbb{F}}_2}) \cong \mathrm{SL}_2(3).$$

In the terminology of [8], we have *maximal monodromy action.*

We can now compute the local L-factor and the conductor exponent of the curve Y over $\mathbb{Q}_2$. It is obvious that the inertial reduction $Z := \bar{Y}/\Gamma$ is a curve of genus zero. This immediately shows that the local L-factor is trivial,

$$L_2(Y,s) = 1.$$

To compute the conductor exponent f_2, we use (2.5) and Corollary 2.14. We get $f_2 = \epsilon + \delta$, where

$$\epsilon = 2g(Y) - 2g(Z) = 2$$

and

$$\delta = \int_0^\infty (2g(Y) - 2g(\bar{Y}^u))\mathrm{d}u,$$

where $\Gamma^u \subset \Gamma$ are the higher ramification groups and $\bar{Y}^u := \bar{Y}/\Gamma^u$.

From the Database of Local Fields ([7]) we known that the highest upper jump of Γ is $u = 1/2$ (i.e. $\Gamma^u = 0$ for $u \geq 1/2$). For $u < 1/2$, the groups Γ^u contains the element σ and hence $g(\bar{Y}_u) = 0$. It follows that

$$\delta = \int_0^2 \frac{1}{2}\mathrm{d}u = 1$$

and hence $f_2 = 2 + 1 = 3$. This agrees with the value for the conductor of Y which, according to `Sage`, is

$$c(E) = 2^3 \cdot 59.$$

4.2 The étale locus

Let L/K be – as before – a finite Galois extension over which Y_L has semistable reduction and let $\mathcal{Y}$ be a semistable model of Y_L. After enlarging L/K, if necessary, we may further assume the following:

(i) all ramification points of the cover $\phi_L : Y_L \to X_L$ specialize to smooth points of $\mathcal{Y}_s$,

(ii) $\mathcal{Y}$ is the minimal semistable model of Y_L with property (i).

It is easy to see that a model $\mathcal{Y}$ with properties (i) and (ii) exists (for L sufficiently large) and is unique. We call it the *canonical semistable model* of Y_L (with respect to ϕ). This is NOT the same as the stably marked model we defined in § 3: the ramification points specialize to pairwise distinct points of the stably marked model, but this is not the case for the canonical semistable model. We write $\bar{Y} := \mathcal{Y}_s$ for the special fiber of $\mathcal{Y}$.

Recall that we also assume that L contains a pth root of unity ζ_p. This implies that $\phi_L : Y_L \to X_L$ is a Galois cover, with cyclic Galois group G of order p. The uniqueness of the model $\mathcal{Y}$ shows that the G-action on Y_L extends to $\mathcal{Y}$. We set $\mathcal{X} := \mathcal{Y}/G$. Then $\mathcal{X}$ is a semistable model of X_L, and the quotient map $\mathcal{Y} \to \mathcal{X}$ is finite. Therefore, $\mathcal{Y}$ is the normalization of $\mathcal{X}$ with respect to Y_L. This means that (at least in principle) it is enough to determine $\mathcal{X}$ in order to determine $\mathcal{Y}$. We also call $\mathcal{X}$ the *canonical semistable model* of X_L (with respect to ϕ). We write $\bar{X} := \mathcal{X}_s$ for the its special fiber and obtain a finite map $\bar{\phi} : \bar{Y} \to \bar{X}$ called the *(canonical) semistable reduction* of ϕ.

Definition 4.4 Let $\bar{U}^{\rm et} \subset \bar{X}$ denote the open subset over which $\bar{\phi} : \bar{Y} \to \bar{X}$ is étale and let

$$U_L^{\rm et} := \mathrm{sp}_{\mathcal{X}}^{-1}(\bar{U}^{\rm et}) \subset X_L^{\rm an}$$

be its inverse image under the specialization map. We call $U_L^{\rm et}$ the *étale locus*.

Example 4.5 Assume that $p = 2$ and that f has degree 3 and no multiple roots,

$$Y : y^2 = f(x) = x^3 + ax^2 + bx + c.$$

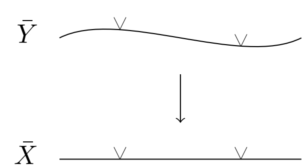
Then Y is an elliptic curve ($g = 1$), and the branch locus of ϕ consists of ∞ and the three zeroes of f.

Let $\mathcal{Y} \to \mathcal{X}$ be the canonical semistable model of ϕ and $\bar{\phi} : \bar{Y} \to \bar{X}$ its reduction. The following three cases may occur:

(i) **(potentially good ordinary reduction)** $\bar{Y}$ and $\bar{X}$ are smooth projective curves (of genus 1 and 0) and $\bar{\phi} : \bar{Y} \to \bar{X} \cong \mathbb{P}^1_{\mathbb{F}_L}$ is a Galois cover of degree 2 branched at two points. The curve $\bar{Y}$ is given generically by an equation of the form

$$z^2 + tz = 1 + t^3,$$

where t is a suitable parameter on $\bar{X} \cong \mathbb{P}^1_{\mathbb{F}_L}$. Hence $\bar{Y}$ is an ordinary elliptic curve over $\mathbb{F}_L$.



In the picture, the ticks indicate a point where two branch points specialize to $\bar{X}$ (resp. where two ramification points specialize to $\bar{Y}$). These points are also the branch points (resp. ramification points) of the cover $\bar{\phi}$. Hence $\bar{U}^{\rm et} \cong \mathbb{P}^1_{\mathbb{F}_l} \backslash \{0, \infty\}$. The étale locus is a closed 'thin' annulus

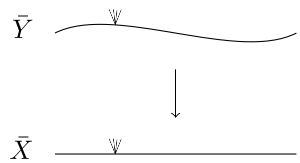
$$U_L^{\rm et} = \{t \mid v_L(t) = 0\},$$

where t is any lift to $X_L^{\rm an}$ of the coordinate t for $\bar{X}$.

(ii) **(potentially good supersingular reduction)** $\bar{Y}$ and $\bar{X}$ are smooth (of genus 1 and 0) and $\bar{\phi} : \bar{Y} \to \bar{X} \cong \mathbb{P}^1_{\mathbb{F}_L}$ is a Galois cover of degree 2 branched at one points. The curve $\bar{Y}$ is given generically by an equation of the form

$$z^2 + z = t^3,$$

where t is a suitable parameter on $\bar{X} \cong \mathbb{P}^1_{\mathbb{F}_L}$. Hence $\bar{Y}$ is a supersingular elliptic curve over $\mathbb{F}_L$.



As in the previous pictures, small ticks indicate specializations of branch resp. ramification points of ϕ. Here all four branch points (resp. ramification points) specialize to the unique ramification point (resp. branch point) of $\bar{\phi}$. Hence $\bar{U}^{\mathrm{et}} \cong \mathbb{A}^1_{\mathbb{F}_L}$ and the étale locus is a closed disk, of the form

$$U_L^{\mathrm{et}} = \{t \mid v_L(t) \leq 1\},$$

where t is a lift of the coordinate t for $\bar{X}$ to X_L^{an}. This is exactly the situation in Example 4.2.

(iii) **(multiplicative reduction)** The curve $\bar{X}$ is the union of two smooth components $\bar{X}_1, \bar{X}_2$ of genus zero, intersecting transversely in a unique point $\bar{x}$. The curve $\bar{Y}$ is the union of two smooth components of genus zero, intersecting in the two points $\bar{y}_1, \bar{y}_2$ lying over $\bar{x}$. The map $\bar{\phi}$ restricts to Galois covers $\bar{Y}_i \to \bar{X}_i$ of degree 2 ramified over one point, unramified over $\bar{x}$. We can choose coordinates t_i for $\bar{X}_i \cong \mathbb{P}^1_{\mathbb{F}_L}$ such that $t_1 t_2 = 0$ is a local equation for $\bar{X}$ at $\bar{x}$ and such that the maps $\bar{Y}_i \to \bar{X}_i$ are given by the equations

$$z_i^2 + z_i = t_i, \quad i = 1, 2.$$

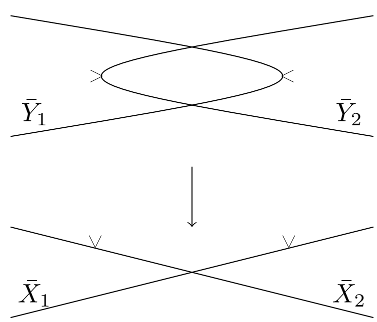


Figure 4.3: $\phi : \bar{Y} \to \bar{X}$ in case of multiplicative reduction

As in the first case, the four branch points of ϕ specialize in two pairs to the two branch points of $\bar{\phi}$, which are given by $t_1 = \infty$ on $\bar{X}_1$ and $t_2 = \infty$ on $\bar{X}_2$. Hence the étale locus is a 'thick' closed annulus of the form

$$U_L^{\mathrm{et}} = \{t \mid 0 \leq v_L(t_1) \leq \epsilon\},$$

where t_1 denote a lift of the coordinate t_1 on $\bar{X}_1$ to $X_L^{\rm an}$ and $\epsilon > 0$ is a rational number representing the 'thickness' of the annulus. If $\beta \in L$ is any element with $v_L(\beta) = \epsilon$, then $t_2 = \beta t_1^{-1}$ is a lift of the coordinate t_2 on $\bar{X}_2$ to $X_L^{\rm an}$.

Proposition 4.6 (i) *The open subset $\bar{U}^{\rm et}$ is nonempty and affine. In particular, it does not contain any irreducible components of $\bar{X}$.*

(ii) *The subset $U_L^{\rm et} \subset X^{\rm an}$ is an affinoid subdomain, with canonical reduction $\bar{U}^{\rm et}$.*

(iii) *The étale locus $U_L^{\rm et}$ descends to an affinoid subdomain $U^{\rm et} \subset X$ such that $U_L^{\rm et} = U^{\rm et} \otimes_K L$, independent of the extension L/K.*

Proof: Easy. □

The following theorem shows that the étale locus $U^{\rm et} \subset X^{\rm an}$ determines to a large extend the stable reduction of Y, including the extension L/K over which we obtain semistable reduction.

Theorem 4.7 *Let L/K be a finite extension of K, $\mathcal{X}$ a semistable model of X_L and $\mathcal{Y}$ the normalization of $\mathcal{X}$ with respect to ϕ. Assume that the following holds.*

(a) There exists an affine open subset $\bar{U}^{\rm et} \subset \mathcal{X}_s$ such that

$$U_L^{\rm et} = {\rm sp}_{\mathcal{X}}^{-1}(\bar{U}^{\rm et}).$$

(b) The model $\mathcal{X}$ is the minimal semistable model of X_L with property (a).

(c) Let $\bar{Z} \subset \bar{X}$ be an irreducible component whose intersection with $\bar{U}^{\rm et}$ is nonempty. Then for some L-rational point $x_0 \in X(L)$ specializing to a point on $\bar{Z} \cap \bar{U}^{\rm et}$, the fiber $\phi^{-1}(x_0)$ consists entirely of L-rational points.

Then $\mathcal{Y}$ is the canonical semistable model of Y_L.

Proof: See [12]. □

The main result of [12] shows that the étale locus $U^{\rm et} \subset X^{\rm an}$ is given by explicit inequalities between absolute values of certain polynomials depending on f. Combined with Theorem 4.7 this result can be used to compute the semistable reduction of Y in practice. To formulate the result (which is done in Theorem 4.12 below), the following definition is useful.

Definition 4.8 A connected component $\bar{D}$ of $\bar{U}^{\rm et}$ which is isomorphic to the affine line $\mathbb{A}^1_{\mathbb{F}_l}$ is called a *tail*. The corresponding connected component of $U_L^{\rm et}$,

$$D := {\rm sp}_{\mathcal{X}}^{-1}(\bar{D}) \subset X_L^{\rm an},$$

is called a *tail disk*. The union of all tail disks is denoted by $U_L^{\rm tail}$, its complement

$$U_L^{\rm int} := U_L^{\rm et} \backslash U_L^{\rm tail}$$

is called the *interior étale locus*.

Let us fix, for the moment, a point $\alpha \in X^{\rm an}\backslash B_\phi$. It is no restriction to assume that α lies in the closed unit disk (with respect to the coordinate x). Using x to identify X_L with $\mathbb{P}^1_L$, this means that $\alpha \in \mathcal{O}_L$, where L/K is a sufficiently large finite extension of K.

For any nonnegative real number r we consider the closed disk

$$D_r = \{x \mid v_L(x-\alpha) \geq r\}$$

with center α and 'radius' r. If $r \in \mathbb{Q}$ is rational then this is an affinoid subdomain of $X_L^{\rm an}$.[1] Furthermore, D_r is isomorphic to the closed unit disk if and only if $r \in v_L(L^\times)$. Note that D_r gets smaller as the radius r increases. Since $\phi : Y \to X$ is finite, the inverse image $C_r := \phi^{-1}(D_r) \subset Y_L^{\rm an}$ is an affinoid subdomain for $r \in \mathbb{Q}$. We say that D_r *splits* if, after replacing L be a finite extension, C_r is the disjoint union of p disjoint closed disks which are mapped isomorphically onto D_r.

We define two real numbers $\lambda = \lambda(\alpha)$ and $\mu = \mu(\alpha)$ which depend on α:

$$\lambda := \inf\{r \mid D_r \text{ splits}\},$$

and

$$\mu := \inf\{r \mid D_r \cap B_\phi = \emptyset\}.$$

The following statements are either clear or relatively easy to prove.

Proposition 4.9 (i) *We have*

$$\mu = \max_i v_K(\alpha - \alpha_i) \in \mathbb{Q}.$$

(Recall that α_i are the zeroes of the polynomial f.) In particular, D_μ is an affinoid subdomain, and it contains at least one zero of f.

(ii) *We have $\lambda \in \mathbb{Q}$. The disk D_λ is the smallest of all disks D_r which does not split.*

(iii) *We have $\lambda \geq \mu$.*

(iv) *The point α lies in $U^{\rm int}$ if and only if $\mu = \lambda$.*

Part (i) of Proposition 4.9 shows that μ, as a function of α, is 'nice', i.e. can be expressed as a linear form in the valuation of an analytic function on X, evaluated in α. Our next goal is to give a similar formula for λ. As a result we obtain, using Proposition 4.9 (iv), explicit equations for the affinoid $U^{\rm int} \subset X$. Actually, we will also get explicit inequalities describing $U^{\rm tail} \subset X$ (Theorem 4.12 below).

The trick is to look at the Taylor expansion of $f/f(\alpha)$ at $x = \alpha$, i.e. we substitute $x = \alpha + t$ and write:

$$f(\alpha + t) = f(\alpha)\Big(1 + \sum_{i=1}^n a_i(\alpha) t^i\Big),$$

[1] In Berkovich's theory, D_r is affinoid for all r, and *strictly affinoid* if $r \in \mathbb{Q}$.

where

$$a_i = \frac{f^{(i)}}{i!f} \in K(x)$$

and where n is the degree of f. We choose an integer m such that $1 \leq m \leq n/p$.

Lemma 4.10 *There exists unique polynomials $H, G \in K(x)[t]$ of the form*

$$H = 1 + \sum_{i=1}^{m} b_i t^i, \quad G = \sum_{k=m+1} c_k t^k,$$

with $b_i, c_k \in K(x)$ such that

$$F := 1 + \sum_{i=1}^{n} a_i t^i = H^p + G. \tag{4.14}$$

Proof: (Compare with (4.9) – (4.11)) Equation (4.14) gives rise to a system of linear equations in the b_i in row echelon form, with a unique solution over the field $K(x)$. This determines H uniquely, and then $H := F - H^p$ is uniquely determined as well. □

Remark 4.11 The coefficients c_k of H are of the form

$$c_k = \frac{d_k}{f^k},$$

with $d_k \in K[x]$. See (4.12).

We can now formulate our main result, which gives an explicit description of the affinoids U^{int} and U^{tail}. Set

$$\tilde{\lambda} := \max_{m+1 \leq k \leq n} \frac{p/(p-1) - v_K(c_k(\alpha))}{k}. \tag{4.15}$$

Also, let $S \subset \{m+1, \ldots, n\}$ be the subset of all k where the maximum above is achieved.

Theorem 4.12 *Assume that $m = \lfloor n/p \rfloor$.*

(i) *We have*

$$\lambda = \max\{\mu, \tilde{\lambda}\}.$$

(ii) *The point α lies in U^{int} if and only if $\tilde{\lambda} \leq \mu$.*

(iii) *The point α lies in U^{tail} if and only if both of the following conditions hold:*

   (a) *$\tilde{\lambda} > \mu$,*

   (b) *S contains an element k which is not a power of p.*

Remark 4.13 (i) It follows from the choice of $m = \lfloor n/p \rfloor$ that the set S does not contain two elements of the form $k, p^s k$, with $s > 0$.

(ii) Moreover, there exists a unique power of p with $m+1 \leq p^l \leq n$. Therefore, Condition in (i) (b) is equivalent to $S \neq \{p^l\}$.

(iii) The condition $m = \lfloor n/p \rfloor$ can often be replaced by a weaker one, see [12], ??.

Proof: (of Theorem 4.12) We will only sketch the proof. For full details, see [12]. For the proof we may assume that $x \notin U^{\mathrm{int}}$. By Proposition 4.9 (iv) this means that $\lambda < \mu$. Set

$$h := 1 + \sum_{i=1}^{m} b_i(\alpha) t^i, \quad g := \sum_{k=m+1}^{n} c_k(\alpha) t^k.$$

Then (4.14) specializes to

$$f = f(\alpha)(h^p + g). \tag{4.16}$$

For $r \geq 0$ let $v_r : L(t)^\times \to \mathbb{Q}$ denote the 'Gauss' valuation with $v_r(t) = r$. Then

$$v_r(g) = \min_k \Big(v_L(c_k(\alpha)) + rk \Big).$$

It follows from the definition of $\tilde{\lambda}$ that

$$v_r(g) \geq \frac{p}{p-1} \quad \Leftrightarrow \quad r \geq \tilde{\lambda}, \tag{4.17}$$

and that equality holds on the left hand side if and only if it holds on the right hand side. Together with (4.17), [12], Proposition 4.7, shows that $\lambda \geq \tilde{\lambda}$. Moreover, it follows from [12], Proposition 4.4, that

$$v_r(h-1) \geq 0 \quad \forall\, r \geq \lambda \tag{4.18}$$

and that

$$v_\lambda(h) = v_{\tilde{\lambda}}(h) = 0. \tag{4.19}$$

Let us choose the extension L/K such that the following holds:

- $\alpha \in L$.
- There exists elements $\beta, \gamma \in L$ with $v_L(\beta) = \lambda$ and $v_L(\gamma) = 1/(p-1)$. (To make Equation (4.21) below look nicer, we also assume that $p\gamma^{1-p} \equiv -1 \pmod{\mathfrak{m}_L}$. This is no restriction.)
- There exists a pth root $\delta = f(\alpha)^{1/p} \in L$ of $f(\alpha)$.

Consider the substitutions

$$x = \alpha + \beta x_1, \quad y = \delta(h + \gamma w).$$

Using (4.16) we see that they transform the equation $y^p = f(x)$ into an equation for w of the form

$$w^p + \ldots + p\gamma^{1-p}h^{p-1}w = \gamma^{-p}g. \tag{4.20}$$

We consider the coefficients in this equation as polynomials in x_1 over L. Using (4.17) and (4.18) one checks that all these polynomials have themselves integral coefficient. Moreover, reducing (4.20) modulo $\mathfrak{m}_L$ gives the equation

$$w^p - w = \bar{g} = \sum_{k\in S} \bar{c}_k x_1^k, \tag{4.21}$$

with $\bar{c}_k \in \mathbb{F}_L^\times$, and where $S \subset \{m+1, \ldots, n\}$ is the set defined after (4.15). Now (4.21) is an Artin-Schreier equation which defines a connected étale cover of degree p of the affine line $\mathbb{A}^1_{\mathbb{F}_L}$ with coordinate x_1. Let

$$\bar{Y}_1 \to \bar{X}_1 \cong \mathbb{P}^1_{\mathbb{F}_L}$$

denote the corresponding cover between smooth projective curves, and let $\bar{Y}_1^\circ \subset \bar{Y}_1$ and $\bar{X}_1^\circ \subset \bar{X}_1$ denote the affine parts with coordinates x_1 and w. Then $\bar{X}_1^\circ$ is the canonical reduction of the closed disk $D_{\tilde{\lambda}}$. It follows that $\bar{Y}_1^\circ$ is the canonical reduction of $C_{\tilde{\lambda}} = \phi^{-1}(D_{\tilde{\lambda}})$. Using (4.21) and Remark 4.13 (i) it is easy to see that $C_{\tilde{\lambda}}$ is connected, i.e. the disk $D_{\tilde{\lambda}}$ does not split. On the other hand, since $\bar{Y}_1^\circ \to \bar{X}_1^\circ$ is étale, every disk D_r with $r > \tilde{\lambda}$ splits. It follows that $\tilde{\lambda} = \lambda > \mu$. Therefore, Condition (a) is always true if $x \notin U^{\rm int}$. So to prove (iii) we must show that $x \in U^{\rm tail}$ if and only if (b) holds. Moreover, we have also shown that (i) and (ii) hold.

By Remark 4.13 (ii), Condition (b) in (iii) is equivalent to the condition that S is not of the form $S = \{p^l\}$. It is easy to see from (4.21) that this latter condition holds if and only $g(\bar{Y}_1) > 0$. It is also easy to see that this is equivalent to the condition $\alpha \in U^{\rm tail}$. This completes the proof of the Theorem. □

Example 4.14 Let us consider the elliptic curve

$$Y:\ y^2 = f(x) := 32 + x^2 + 2x^3$$

over $K = \mathbb{Q}_2$. We use Theorem 4.12 to first compute the étale locus $U^{\rm et} \subset X^{\rm an}$ and then the semistable reduction of Y. It is natural to start with computing the interior part $U^{\rm int}$ of $U^{\rm et}$. By Theorem 4.12 (ii),

$$U^{\rm int} = \{x \mid \tilde{\lambda}(x) \leq \mu(x)\}. \tag{4.22}$$

Here the notation $\tilde{\lambda}(x), \mu(x)$ means that we consider $\tilde{\lambda}, \mu$ as functions on $X^{\rm an}$, expressed in terms of the coordinate x. To simplify the evaluation of the functions $\tilde{\lambda}(x)$ and $\mu(x)$, it is also good idea to restrict attention to certain carefully chosen subsets of $X^{\rm an}$.

The Newton polygon of $f \in K[x]$ tells us that the roots of f have valuation $-1, 5/2, 5/2$. Therefore, we consider the open annulus

$$A := \{x \mid -1 < v_K(x) < 5/2\}$$

and denote the function $v_K(x)$ on A simply by r. It follows from Proposition 4.9 (i) that the restriction of the function μ to A is given by the simple formula

$$\mu(x) = r. \tag{4.23}$$

Morever, the valuation of $v_K(f(x))$, considered as a function on A, has the simple form

$$v_K(f(x)) = 2r. \tag{4.24}$$

To evaluate the function $\tilde{\lambda}$, we need to compute the rational functions

$$c_k(x) = \frac{d_k(x)}{f(x)^k}, \quad k = 2, 3.$$

We find

$$d_2 = 3x^4 + 2x^3 + 192x + 32, \quad d_3 = 2f(x)^2. \tag{4.25}$$

Inspection of the Newton polygon of d_2 shows that

$$v_K(d_2(x)) \geq \begin{cases} 4r, & -1 < r \leq 1, \\ 1 + 3r, & 1 \leq r \leq 4/3, \\ 5, & r \geq 4/3, \end{cases} \tag{4.26}$$

and that equality holds in (4.26) for $r \neq 1, 4/3$. Combining (4.24), (4.25) and (4.26), we see that

$$v_K(c_2(x)) \geq \begin{cases} 0, & -1 < r \leq 1, \\ 1 - r, & 1 \leq 1 \leq 4/3, \\ 5 - 4r, & r \geq 4/3, \end{cases} \tag{4.27}$$

with equality for $r \neq 1, 4/3$. With similar but simpler arguments we see that

$$v_K(c_3(x)) = 1 - 4r. \tag{4.28}$$

The function $\tilde{\lambda}(x)$ was defined as

$$\tilde{\lambda}(x) = \max\Big(\frac{2 - v_K(c_2(x))}{2}, \frac{2 - v_K(c_3(x))}{3}\Big). \tag{4.29}$$

Hence it follows from (4.22) and (4.23) that

$$\begin{aligned} x \in U^{\mathrm{int}} \quad &\Leftrightarrow \quad \tilde{\lambda}(x) \leq \mu(x) \\ &\Leftrightarrow \quad v_K(c_2(x)) \geq 2 - 2r \quad \text{and} \quad v_K(c_3(x)) \geq 2 - 3r. \end{aligned} \tag{4.30}$$

By (4.27), the first condition $v_K(c_2(x)) \geq 2 - 2r$ holds if and only if $1 \leq r \leq 3/2$. On the other hand, by (4.28) the second condition $v_K(c_3(x)) \geq 2 - 3r$ holds if and only if $r \geq 1$. We conclude that $U^{\mathrm{int}} \cap A$ is the closed annulus

$$A_1 := \{x \mid 1 \leq v_K(x) \leq 3/2\}.$$

We claim that $U^{\mathrm{et}} = A_1$ (this means that $U^{\mathrm{et}} \subset A$, and $U^{\mathrm{tail}} = \emptyset$). To prove this, it suffices to show that the closed annulus A_1 already determines the stable reduction of Y (more precisely, it satsifies the conditions in Theorem 4.7). So let L/K be a finite Galois extension containing a square root of 2. Then there exists a semistable model $\mathcal{X}$ of $X_L = \mathbb{P}^1_L$ which is minimal with the property that there exists an affine open subset $\bar{A}_1 \subset \bar{X} := \mathcal{X}_s$ such that $A_1 =]\bar{A}_1[_{\mathcal{X}}$. In the language of §3, the model $\mathcal{X}$ corresponds to the set of equivalence classes of coordinates given by x_1, x_2, where

$$x = 2x_1, \quad x = 2^{3/2}x_2.$$

This means that the special fiber $\bar{X}$ of $\mathcal{X}$ is the union of two irreducible components $\bar{X}_1, \bar{X}_2$ intersecting transversally in a unique point $\bar{x}$. The coordinates x_1, x_2 induce isomorphisms $\bar{X}_i \xrightarrow{\sim} \mathbb{P}^1_{\mathbb{F}_L}$ sending the point of intersection $\bar{x}$ to the point $x_1 = 0$ on $\bar{X}_1$ and to the point $x_2 = \infty$ on $\bar{X}_2$.

Now let $\mathcal{Y}$ denote the normalization of $\mathcal{X}$ in Y_L. If the extension L/K is sufficiently large, then it follows from Theorem 4.7 that $\mathcal{Y}$ is the canonical semistable model of Y. Also, we are in Case (iii) of Example 4.5. Therefore, the special fiber $\bar{Y}$ of $\mathcal{Y}$ consists of two smooth irreducible components $\bar{Y}_1, \bar{Y}_2$ of genus 0 which intersect transversally in two distinct points $\bar{Y}_1, \bar{Y}_1$. The map $\bar{\phi} : \bar{Y} \to \bar{X}$ maps $\bar{Y}_1$ to $\bar{X}_1$ and $\bar{Y}_2$ to $\bar{X}_2$. The maps $\bar{Y}_i \to \bar{X}_i$ are generically étale finite covers of degree 2, ramified in precisely one point, namely the point $x_1 = \infty$ on $\bar{X}_1$ and $x_2 = 0$ on $\bar{X}_2$. See Figure 4.3.

To check this explicitly, and to nail down the extension L/K, we first substitute $x = 2x_1$ into f. This allows us to write f in the form

$$f(x) = 4\big(x_1^2 + 4(x_1^3 + 1)\big).$$

Therefore, after subsituting $y = 2(x_1 + 2w_1)$ into the equation $y^2 = f$ we arrive, after the usual computation, at the equation

$$w_1^2 + x_1 w_1 = x_1^3 + 2. \tag{4.31}$$

Thsi equation describes the 'naive model' of $\mathcal{Y}$ with respect to the coordinate x_1. It special fiber is the semistable curve of genus 1 with equation

$$w_1^2 + x_1 w_1 = x_1^3. \tag{4.32}$$

However, this equation does not correctly describe the component $\bar{Y}_1$ of $\bar{Y}$ because of the singularty at the point $(x_1, w_1) = (0, 0)$. To obtain the 'correct equation' we can do the substitution $w_1 = x_1 z_1$, yielding the Artin-Schreier

$$z_1^2 + z_1 = x_1.$$

Similarly, we can check how the component $\bar{Y}_2$ arises. We substitute $x = 2^{3/2}x_2$ into f and write f as

$$f = 2^3\big(x_2^2 + 4(1 + 2^{1/2}x_2^3)\big).$$

This leads us to substitute $y = 2^{3/2}(x_2 + 2w_2)$ into the equation $y^2 = f$. We obtain the equation

$$w_2^2 + x_2 w_2 = 1 + 2^{1/2} x_2^3$$

which reduces to the equation

$$w_2^2 + x_2 w_2 = 1$$

over $\mathbb{F}_L$.

Remark 4.15 Already in this really simple example, the analysis of the inequalities defining the étale locus is quite tricky, althought the end result is relatively easy to state, and could have been guess by a sharp glance at the polynomial f. In general, we need a solution to the following problem to turn our result (Theorem 4.7 and Theorem 4.12) into a true algorithm.

Problem 4.16 Let K be a p-adic number field and $X := \mathbb{P}^1_K$. Given an affinoid subdomain $U \subset X^{\rm an}$, defined by explicit inequalities between valuations of rational functions on X, find a finite extension L/K and a semistable model $\mathcal{X}$ of X_L such that $U =]\bar{U}[_{\mathcal{X}}$ for an open affine subset $\bar{U} \subset \mathcal{X}_s$.

We plan to work out a general algorithm solving Problem 4.16 and implementing it in `Sage`. The general ideal is to use the 'tree structure' of the Berkovic space $X^{\rm Berk}$ and the theory of MacLane valuations (see [11]).

4.3 Equidistant branch locus

Under a certain assumption on the set of branch points of ϕ our main result becomes much simpler, and we recover the earlier results of Lehr and Matignon ([10], [8]).

Definition 4.17 We say that the cover $\phi : Y \to X$ has *equidistant branch locus* if, for some finite extension L/K and for choice of the isomorphism $X_L \cong \mathbb{P}^1_L$ (corresponding to a coordinate x), the branch divisor D_L of ϕ extends to a relative divisor $\mathcal{D} \subset \mathcal{X}_0$ of the smooth model $\mathcal{X}_0 := \mathbb{P}^1_{\mathcal{O}_L}$ which is finite and étale over $\operatorname{Spec} \mathcal{O}_L$.

In the terminology introduced at the end of §3.1 this means that the stably marked model of (X_L, D_L) is smooth. In plain words, the condition in Definition 4.17 means the following. Let L'/L be the splitting field of f over L and

$$f = c(x - \alpha_1)^{a_1} \cdot \ldots \cdot (x - \alpha_r)^{a_r}$$

with $\alpha_i \in L'$, then there exists a fractional linear transformation $\sigma : \mathbb{P}^1_L \xrightarrow{\sim} \mathbb{P}^1_L$ such that $\alpha_i' := \sigma(\alpha_i) \in \mathcal{O}_{L'}$ is integral and $\alpha_i' - \alpha_j' \in \mathcal{O}_{L'}^\times$ is a unit, for all $i \neq j$. Clearly, this condition is satisfied if $f \in \mathcal{O}_K[x]$ is integral and the discriminant $d(\tilde{f}) \in \mathcal{O}_K^\times$ is a unit, where $\tilde{f} := f/(f, f')$ is the radical of f (and then we may take $L := K$ and $\sigma = \mathrm{Id}_X$).

Theorem 4.18 *Assume that $\phi : Y \to X$ has equidistant branch locus. Then $U^{\rm et} = U^{\rm tail}$, i.e. $U^{\rm et}$ becomes (after a finite extension of K) a finite union of closed disks.*

Assume, moreover, that f is integral and monic. Then the following holds:

(i) *The étale locus $U^{\rm et}$ is contained in the closed unit disk*

$$\{x \mid v_K(x) \geq 0\}.$$

(ii) *Let n, m, p^l be as in Theorem 4.12 and Remark 4.13. Let $\Delta := d_{p^l}(x)$ be the monodromy polynomial. Then every root of Δ (over the algebraic closure $K^{\rm ac}$) lies in one of the tail disks (i.e. the connected components of $U^{\rm et}_{K^{\rm ac}}$), and every tail disk contains one of the roots of Δ.*

(iii) *Let L/K be a Galois extension which satifies the following conditions:*

  (a) *The étale locus $U^{\rm et}_L$ is the finite union of closed unit disks $D_1, \ldots, D_s$. Therefore, there exists $\alpha_i, \beta_i \in L$ such that*

$$D_i = \{x \mid v_L(x - \alpha_i) \geq v_L(\beta_i)\},$$

  for $i = 1, \ldots, s$.

  (b) *For all i there exists $\gamma_i \in L$ with $\gamma_i^p = f(\alpha_i)$.*

  Then Y has semistable reduction over L.

Together with the explicit recipe for computing tail disks and the corresponding equations for the tail components in the proof of Theorem 4.12, Theorem 4.18 gives a rather straightforward algorithm for computing the stable reduction of Y in the case of equidistant branch locus. We will illustrate this in one explicit example below (Example 4.19).

Example 4.19 Let us look at the genus 2 curve

$$Y : \; y^2 = f(x) := x^5 + x^3 + 3$$

from Example 3.10. There we analyzed the stable reduction of Y at the tame bad primes $p = 3, 3137$. Now we let $p = 2$ and consider Y as a curve over $K = \mathbb{Q}_2$.

The reduction of f' is prime to the reduction of f and has two irreducibles factors:

$$\bar{f}' = x^4 + x^2 = x^2(x+1)^2.$$

The first thing we see from this is that the branch locus is equidistant. Therefore, by Theorem 4.18, the étale locus consists only of tail disks. Moreover, all tail disks lie in one of the two residue disks

$$D^\circ(0) = \{x \mid v_K(x) > 0\}, \quad D^\circ(1) = \{x \mid v_K(x-1) > 0\},$$

and both of these residue disks contain at least one tail disk. Since the genus of Y is $g = 2$ and each tail disk contributes to the genus by a positive integer, it follows that each residue disk $D^\circ(0)$ and $D^\circ(1)$ contains exactly one tail disk, and each of these tail disks corresponds to a tail component of genus 1 in the stable reduction of Y. So without any further computation we know already what the stable reduction $\bar{\phi} : \bar{Y} \to \bar{X}$ looks like:

picture

In fact, we also know that the tail components $\bar{Y}_1$ and $\bar{Y}_2$ are both isomorphic (as curves over the finite field $\mathbb{F}_L$, which is some finite extension of $\mathbb{F}_2$) to the supersingular elliptic curve with equation

$$z^2 + z = x^3.$$

However, this tells us very little about the local L-factor and the conductor exponent. What we really need to know is a Galois extension L/K over which the stable reduction occurs, and the action of $\Gamma = \mathrm{Gal}(L/K)$ on $\bar{Y}$.

Let D_1 (resp. D_2) denote the tail disks corresponding to the components $\bar{X}_1$ and $\bar{X}_2$ of $\bar{X}$. Then D_1 (resp. D_2) lies in the residue disk $D^\circ(0)$ (resp. $D^\circ(1)$). To determine D_1 and D_2 we compute the monodromy polynomial $\Delta = d_4$ (we set $m := \lfloor 5/2 \rfloor = 2$ and then 4 is the unique power of 2 in $\{3, 4, 5\}$; see ??). We find

$$\begin{aligned}\Delta = 2^{-6}\big(&95\,x^{16} + 300\,x^{14} + 386\,x^{12} - 720\,x^{11} + 188\,x^{10} - 600\,x^9 \\ &- 9\,x^8 + 576\,x^7 - 5760\,x^6 - 216\,x^5 - 1296\,x^2 + 8640\,x\big).\end{aligned}$$

We also compute the splitting of Δ into irreducible factors. We find that

$$\Delta = \frac{95}{64}x(x-a)\Delta_1\Delta_2\Delta_3,$$

where $a \in \mathbb{Z}_2$ has valuation $v_K(a) = 2$, and $\Delta_1, \Delta_2, \Delta_3 \in \mathbb{Z}_2[x]$ are irreducible, of degree $6, 4, 4$. The roots of Δ_1 have valuation $2/3$ and hence lie in the first residue disk $D^\circ(0)$. The roots of Δ_2 and Δ_3 have valuation 0 and hence do not lie in $D^\circ(0)$. By Theorem 4.18 (ii), all roots of Δ lie in one of the tail disks D_1 and D_2. Therefore, the points $x = 0, a$ and all roots of Δ_1 are centers of the disk D_1, whereas all roots of Δ_2 and Δ_3 are centers of D_2.

So it is easy to find D_1. We choose the center $\alpha = 0$ and write f in the form

$$f = 3 + x^3 + x^5 = 3(1^2 + g),$$

with

$$g = \frac{1}{3}(x^3 + x^5).$$

Now we look for a variable change $x = \beta x_1$ such that the Gauss valuation of g with respect to x_1 has the value 2. Clearly, this is the case if and only if

$v_K(\beta) = 2/3$. Let L_1/K be the minimal Galois extension such that L_1 contains $3^{1/2}$ and $2^{1/3}$. Applying the variable change

$$x = 2^{2/3}x_1, \quad y = 3^{1/2}(1+2w) \tag{4.33}$$

we obtain, by the usual computation, the new equation

$$w^2 + w = \frac{1}{3}x_1^3 + \frac{2^{4/3}}{3}x_1^5.$$

Over $\mathbb{F}_L$ this equation reduces to

$$w^2 + w = x_1^3, \tag{4.34}$$

which is an equation for the component $\bar{Y}_1$. In particular, we see that the first tail disk is

$$D_1 = \{x \mid v(x) \geq 2/3\}.$$

To find the second tail disk D_2 we need to find a sufficiently good approximation of one of the factors Δ_2 or Δ_3. A computation using so-called *MacLane valuations* (see [11]) shows that

$$\tilde{\Delta}_2 = x^4 + 4\,x^3 + 10\,x^2 + 16\,x + 13$$

is an approximation of Δ_2 with the following property. For every root α of Δ_2 there is a unique root $\tilde{\alpha}$ of $\tilde{\Delta}_2$ such that $v_K(\alpha - \tilde{\alpha}) = 2$. Since $v_L(\alpha - \alpha') = 2/3 < 2$ for two distinct roots α, α' of Δ_2, this shows that all roots of $\tilde{\Delta}_2$ lie in any closed disk which contains all the roots of Δ_2. In particular, any root of $\tilde{\Delta}_2$ must be a center for the tail disk D_2.

Let α be a root of $\tilde{\Delta}_2$, and let L_2/K be the minimal Galois extension containing α and a square root of $f(\alpha) = \alpha^5 + \alpha^3 + 3$. We substitute $x = t + \alpha$ into f and write f in the form

$$\begin{aligned} f &= f(\alpha)\cdot\big(1 + a_1 t + \ldots + a_5 t^5\big) \\ &= f(\alpha)\cdot\big(h^2 + g\big), \end{aligned}$$

with

$$h = 1 + b_1 t + b_2 t^2, \quad g = c_3 t^3 + c_4 t^4 + c_5 t^5$$

and $a_i, b_j, c_k \in L_2$. We now look at the Newton polygons of h and g, which are given by the valuations of its coefficients:

j	0	1	2
$v_{L_2}(a_j)$	0	0	0

k	3	4	5
$v_{L_2}(c_k)$	0	8	0

.

As before, we try to find a substitution $t = \beta x_1$ such that the Gauss valuation of g with respect to x_1 takes the value 2. We see that this is the case if and only if $v_{L_2}(\beta) = 3/2$. We choose $\beta := \alpha - \alpha_2$, where $\alpha_2 \in L_2$ is a root of $\tilde{\Delta}_2$ distinct from α. Performing the variable changes

$$x = \alpha + \beta x_1, \quad y = f(\alpha)^{1/2}\Big(h + 2w\Big)$$

we obtain the new equation

$$w^2 + hw = g,$$

which reduces to the equation

$$w^2 + w = x_1^3$$

over $\mathbb{F}_{L_2}$. This is the equation for the component $\bar{Y}_2$.

We have shown that the curve Y has semistable reduction over the extension $L := L_1L_2/K$, and we have all the information necessary to determine the action of $\Gamma = \mathrm{Gal}(L/K)$ on the stable reduction $\bar{Y}$. Actually, it is enough to look at the action of the two quotient groups $\Gamma_1 := \mathrm{Gal}(L_1/K)$ and $\Gamma_2 := \mathrm{Gal}(L_2/K)$ on the components $\bar{Y}_1$ and $\bar{Y}_2$, respectively. The reason is that the $\Gamma_K = \mathrm{Gal}(\bar{K}/K)$-representation

$$V := H^1_{\mathrm{et}}(Y_{\bar{K}}, \mathbb{Q}_\ell) = V_1 \oplus V_2$$

splits into the direct sum of two subrepresentations V_1, V_2, where V_i is completely determined by the action of Γ_i on the component $\bar{Y}_i$ (see §2; Lemma 2.10 makes a precise, but somewhat weaker statement).

It is easy to see that the extension L_1/K has Galois group

$$\Gamma_1 = S_3 \times C_2$$

and that the inertia group is the unique cyclic subgroup of order 6. The last break in the upper numbering filtration is $u = 1$, with $\Gamma_1^1 = C_2$. Let $\sigma \in \Gamma_1^1$ be the unique nontrivial element. Then $\sigma(3^{1/2}) = -3^{1/2}$ and $\sigma(2^{1/3}) = 2^{1/3}$. Using (4.33) one show by an easy computation that the automorphism induced from σ on the component $\bar{Y}_1$ (which is given by (4.34)) is determined by

$$\sigma(w) = w + 1, \quad \sigma(x_1) = x_1.$$

We conclude that $\bar{Y}_1^1 = \bar{Y}_1/\Gamma_1^1$ has genus zero. This shows that the contribution of the subrepresentation V_1 to the local L-factor $L_2(Y, s)$ is trivial, and its contribution to the Swan conductor is

$$\delta_1 = \int_0^\infty \big(2g(\bar{Y}_1) - 2g(\bar{Y}_1^u)\big)\mathrm{d}u = \int_0^1 2\mathrm{d}u = 2.$$

The Galois group $\Gamma_2 = \mathrm{Gal}(L_2/K)$ of the second extension is found to be isomorphic to $\mathrm{GL}_2(3)$, with inertia group $\mathrm{SL}_2(3)$ (use the DBLF). Moreover, the last break in the higher numbering filtration is $u = 1/2$, and $\Gamma_2^{1/2}$ has order 2 and is equal to the center of Γ_2. In fact, the corresponding subextension is $L_2/L_{2,0}$, where $L_{2,0}$ is the splitting field of $\tilde{\Delta}_2$ (containing α, β) and

$$L_2 = L_{2,0}[f(\alpha)^{1/2}].$$

So if $\sigma \in \Gamma_2^{1/2}$ denotes the unique nontrivial element, then σ fixes α and β, and $\sigma(f(\alpha)^{1/2}) = -f(\alpha)^{1/2}$. Now the same computation as for $\bar{Y}_1$ shows that $\bar{Y}_2^{1/2}$ is a curve of genus zero. Therefore, the contribution of the component $\bar{Y}_2$ to the local L-factor is trivial, and the contribution to the Swan conductor is

$$\delta_2 = \int_0^\infty \big(2g(\bar{Y}_2) - 2g(\bar{Y}_2^u)\big)\mathrm{d}u = \int_0^{1/2} 2\mathrm{d}u = 1.$$

All in all, we see that the local L-factor of Y at $p = 2$ is trivial,

$$L_2(Y, s) = 1,$$

and that the conductor exponent is

$$f_2 = \epsilon + \delta_1 + \delta_2 = 4 + 2 + 1 = 7.$$

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