FUNCTORIALITY

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First, the class of groups we work with are connected reductive algebraic groups, for example $G = \mathrm{GL}_n$. In general, we would work over a global field but today we work over k a number field, and let $\mathbb{A}_k$ be the adeles over k.

Second, we consider automorphic forms: from f, a complex-valued function on the upper half-plane, we define a complex-valued function ϕ_f on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. Letting $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ act, we end up with a representation, so this sits naturally in the space

$$L^2(ZG(k)\backslash G(\mathbb{A}_k), \omega)$$

of square-integrable functions which transform under the center Z according to the character ω. We look inside the space L_0^2 of cusp forms, vanishing at the cusps: let $P = MN \subsetneq G$ be a parabolic subgroup, and insist that

$$\int_{N(k)\backslash N(\mathbb{A})} \phi(ng)\, dn = 0$$

for almost all $g \in G$. An irreducible representation π in L_0^2 decomposes as a tensor product $\pi = \otimes_v \pi_v$ where π_v is a representation of $G(k_v)$, so we need to understand representations of groups over local fields; this is necessary, but not sufficient, as automorphic forms contain some global constraints and not all local representations will necessarily occur. These are often deep questions (e.g., the Ramanujan–Selberg conjecture).

Third, we consider L-groups. Langlands computed constant terms of Eisenstein series and saw a natural product of zeta functions (L-functions), and this led him to the notion of L-groups. Let T be a maximal torus over $\overline{k}$. Let $X = X^*(T)$ be the group of characters of $T(\overline{k})$, and let $X^\vee = X_*(T)$ be the group of cocharacters, maps from $\overline{k}^* \to T(\overline{k})$. Let Σ be the roots of T, the nonzero eigenfunctions of $T(\overline{k})$ on the Lie algebra $\mathfrak{g}(\overline{k})$. Dually, there is also a notion of coroots $\Sigma^\vee$.

From this we have the data $(X, \Sigma, X^\vee, \Sigma^\vee)$. Let $\Delta \subseteq \Sigma$ be the simple roots; then we also have the data $(X, \Delta, X^\vee, \Delta^\vee)$, from which one can recover the group. Taking the dual data $(X^\vee, \Sigma^\vee, X, \Sigma)$, we obtain a corresponding dual group $G^\vee$, a complex group. For example, we can work things out explicitly for $G = \mathrm{GL}_2$, and we find that the dual group gives us back GL_2.

Suppose now that G is semisimple, so G or G_{der} has finite center. Let $C = (\langle \alpha_i, \alpha_j \rangle)$ be the Cartan matrix, where

$$\langle \alpha_i, \alpha_j \rangle = \frac{2\kappa(\alpha_i, \alpha_j)}{\kappa(\alpha_j, \alpha_j)},$$

where κ is the Killing form. The transpose ${}^tC = C^\vee$ is the matrix for the dual data. For $G = \mathrm{Sp}_{2n}$ we have $G^\vee = \mathrm{SO}_{2n+1}$ and vice versa. Over a nonalgebraically closed field, we

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keep track of also the Galois action, and we define ${}^LG = G^\vee \rtimes \Gamma_k$ where $\Gamma_k = \mathrm{Gal}(\overline{k}/k)$, or sometimes replacing Γ_k by the Weil group.

Let R be the root lattice, the $\mathbb{Z}$-lattice generated by the roots in $\mathbb{R}^n$ (n is then the rank of the semisimple group G). Let Q be the weight lattice of all $\chi \in \mathbb{R}^n$ such that $\langle \chi, \alpha \rangle \in \mathbb{Z}$ for all $\alpha \in R$. We have $Q \supseteq X \supseteq R$. If G is simply connected, then $X = Q$; if G is adjoint, then $X = R$.

We then have the following table.

G	$G^\vee$
GL_n	$\mathrm{GL}_n(\mathbb{C})$
SO_{2n}	$\mathrm{SO}_{2n}(\mathbb{C})$
SO_{2n+1}	$\mathrm{Sp}_{2n}(\mathbb{C})$
GSpin_{2n}	$\mathrm{GSO}_{2n}(\mathbb{C})$
GSpin_{2n+1}	$\mathrm{GSp}_{2n}(\mathbb{C})$
$\mathrm{GSpin}_5 = \mathrm{GSp}_4$	$\mathrm{GSp}_4(\mathbb{C})$

The latter is because of an accidental isomorphism.

Now we discuss unramified representations. Let K be a maximal compact subgroup of $G(k_v)$, where $k = k_v$ is a p-adic field. Let T be a maximal torus. By the Iwasawa decomposition, we have $G(k) = T(k)U(k)K$ with U a maximal unipotent subgroup. Let π be an irreducible admissible representation of $G(k)$; we say that π is *unramified* (or *spherical*) if there exists a w in the representation space of π that is fixed by K; in particular, if π is irreducible then there is a unique line going through w that is invariant under $\pi(K)$. If π is unramified, then π is a constituent of $I(\chi)$ (induced representation) from χ a character of $T(k)$, so

$$I(\chi) = \{f(tug) = \chi(t)\delta^{1/2}(t)f(g) : f \text{ is smooth complex-valued on } G\}.$$

If χ is unramified, then χ restricted to ${}^oT(k) = T(k) \cap K$ is identically 1; so unramified characters are homomorphisms from

$$\Lambda = T(k)/{}^oT(k) \to \mathbb{C}^*.$$

Let X_{un} be the group of characters of Λ, and note that $\Lambda = X^\vee(T) = X(T^\vee)$. Then

$$\mathrm{Hom}(\Lambda, \mathbb{C}^*) = \mathrm{Hom}(X(T^\vee), \mathbb{C}^*) = T^\vee.$$

So $T^\vee$ then becomes the main object of study in defining L-functions.

The conclusion: unramified characters are parametrized by $T^\vee$; let W be the Weyl group, acting as $\chi^w(t) = \chi(w^{-1}tw)$, then

$$X_{\mathrm{un}}/W \leftrightarrow T^\vee/W^\vee = G^\vee\text{-conjugacy classes of } T^\vee.$$

Thus $I(\chi)$ and $I(\chi^w)$ have the same constituents.

Langlands realized, by interpreting constant terms of Eisenstein series as products of L-functions, that if r is a finite-dimensional representation of ${}^LG$, and $s \in \mathbb{C}$, and as above we obtain from π_v a conjugacy class $c(\pi_v)$ in $T^\vee/W^\vee$, we can write down the unramified L-function as

$$L(s, \pi_v, r) = \det(1 - r(c(\pi_v))q_v^{-s})^{-1}$$

where q_v is the number of elements in the residue field of k. (Unfortunately, there are a lot more cuspidal representations π of $G(\mathbb{A}_k)$ than there are ρ representations of W_k.)

Let $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ be a cusp form with respect to the classical group $\Gamma_0(N)$ of weight k. Let

$$a_p = p^{(k-1)/2}(\alpha_p + \alpha_p^{-1}).$$

Then f corresponds to a representation $\pi = \otimes_p \pi_p$, and π_p corresponds to the conjugacy class $\begin{pmatrix} \alpha_p & 0 \\ 0 & \alpha_p^{-1} \end{pmatrix}$; if r is the standard representation of $\mathrm{GL}_2(\mathbb{C})$, then the unramified L-function is

$$(1 - \alpha_p p^{-s})^{-1}(1 - \alpha_p^{-1} p^{-s})^{-1}.$$

Now we can define functoriality. Let G, G' be connected reductive groups, where G' is quasisplit. Suppose we have a homomorphism $\phi : {}^L G \to {}^L G'$. Let $\pi = \otimes_v \pi_v$ be a cuspidal representation of $G(\mathbb{A}_k)$; recall that at almost all primes, v, the representation π_v corresponds to conjugacy classes $c(\pi_v)$. Then we can consider the image $\phi(\{c(\pi_v)\})$ of conjugacy classes in ${}^L G'$, and we can ask: do there exist cuspidal automorphic representations $\Pi = \otimes_v \Pi_v$ on $G'(\mathbb{A}_k)$ that agree with these conjugacy classes at almost all v? Whether or not this appears in an L^2-space is a very deep question. Functoriality implies equality of L-functions, root numbers, etc. (We want further to have the second projetion of $\phi(x, w)$ be still w, so we require the map ϕ to be a so-called *L-homomorphism*.)

Most of the time, $G' = \mathrm{GL}_n$; in this case, by strong multiplicity one, if such a representation Π exists, then it is necessarily unique. In other cases, this representation may not be unique, so we group them together into L-packets.

Here are some examples. Symmetric powers of GL_2. Let $G = \mathrm{GL}_2$, and consider

$$\phi = \mathrm{Sym}^m : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_{n+1}(\mathbb{C})$$

defined as follows: if $P(X, Y)$ is a form of degree m and $g \in \mathrm{GL}_2(\mathbb{C})$ then $P_1(X, Y) = P((X, Y)g)$, and expressing coefficients gives you $\mathrm{Sym}^m g$. In this context, the question of functoriality asks: is there a functorial transfer Sym^m of representations?

Theorem. Sym^m *is functorial for* $m = 2$ *(Gelbart–Jacquet, 1978),* $m = 3$ *(Kim–Shahidi, 2002),* $m = 4$ *(Kim, 2002).*

Unfortunately, at this point the image (of the local Galois representations) may not be solvable, so it is likely to be very hard for $m \geq 5$. Each case of functoriality would have many important consequences.

Now we also have the local Langlands correspondence by GL_n by Harris–Taylor, Henniart, Scholze; this allows you to make local candidates, so the hard part is to prove that the corresponding global candidate is indeed automorphic.

Over k_v, the representation π_v gives ρ_v a two-dimensional representation of W'_{k_v}, so we have a homomorphism

$$W_{k_v} \xrightarrow{\rho_v} \mathrm{GL}_2(\mathbb{C}) \xrightarrow{\mathrm{Sym}^m} \mathrm{GL}_{m+1}(\mathbb{C}).$$

and by the local Langlands correspondence, $\mathrm{Sym}^m \pi_v$ corresponds to $\mathrm{Sym}^m \rho_v$. Then the question is whether or not $\otimes_v \mathrm{Sym}^m \pi_v$ is automorphic; and it is known for $m = 2, 3, 4$.

Suppose π comes from a Galois group. Then π corresponds to $\rho : \Gamma \to \mathrm{GL}_2(\mathbb{C})$, and we can postcompose using Sym^m to land in $\mathrm{GL}_{m+1}(\mathbb{C})$. The representation $\mathrm{Sym}^m \rho$ of Γ *does* come from an automorphic representation of $\mathrm{GL}_{m+1}(\mathbb{A})$ for all m (Kim).

For π_v corresponding to the semisimple conjugacy class $\begin{pmatrix} \alpha_v & 0 \\ 0 & \beta_v \end{pmatrix}$ of $\mathrm{GL}_2(\mathbb{C})$, then

$$\mathrm{Sym}^m \begin{pmatrix} \alpha_v & 0 \\ 0 & \beta_v \end{pmatrix} = \mathrm{diag}(\alpha_v^m, \alpha_v^{m-1}\beta_v, \ldots, \beta_v^m).$$

By work of Luo–Rudnick–Sarnak, while the Ramanujan conjecture demands $|\alpha_v| = |\beta_v| = 1$, Kim–Shahidi proved $q_v^{-5/34} < |\alpha_v|, |\beta_v| < q_v^{5/34}$ using Sym^3.

If $\mathfrak{H}$ is the upper half-plane and Γ is a congruence subgroup, inside $L^2(\Gamma\backslash\mathfrak{H})$ consider the eigenvalues of the Laplace operator $\Delta = -y^2(d^2/dx^2 + d^2/dy^2)$; Selberg conjectured that the smallest nonzero eigenvalue satisfies

$$1/4 \leq \lambda_1(\Gamma\backslash\mathfrak{H});$$

the best known result is due to Blomer–Brumely 2012 and Kim–Sarnak 2002 over $\mathbb{Q}$ that

$$\lambda_1(\Gamma\backslash\mathfrak{H}) \geq 1/4 - (7/64)^2 = 0.238;$$

this already leads to many nice results. And we have

$$q_v^{-7/64} < |\alpha_v|, |\beta_v| < q_v^{7/64}.$$

This was all just for GL_2. Now let G be a group whose L-group is classical, so $G = \mathrm{SO}, \mathrm{Sp}, \mathrm{U}, \mathrm{GSpin}$. Then ${}^L G \hookrightarrow \mathrm{GL}_N(\mathbb{C})$, and functoriality is proven in these cases. Using the trace formula, these have been proven by Arthur for the first two $G = \mathrm{SO}, \mathrm{Sp}$, and by Mok and others for $G = \mathrm{U}$. This has consequence for the entirety of the corresponding L-functions: for example, if π is a repressentation of Sp_4, then the L-functions $L^S(s, \pi, \mathrm{spin})$ of degree 4 is entire and $L^S(s, \pi, \mathrm{std})$ of degree 5 has poles only at $s = 1$, and this extends to several symmetric powers as well.

The Langlands "Beyond Endoscopy" philosophy indicates that we should use the trace formula to further investigate functoriality.