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Quantum Information Science

March 4, 2008

Problem Set #3

(due in class, 13-Mar-08)

Lecture Topics (3/4, 3/6, 3/11): cluster state and measurement based quantum computation, topological quantum field theory = QC, adiabatic quantum computation and the grover search

Recommended Reading: Nielsen and Chuang 10.6

Problems:

P1: (**Toric code**) Consider $2\ell^2$ qubits each residing at an edge of an $\ell \times \ell$ square lattice that is drawn on a 2-dimensional torus. As in our last problem set, define plaquette operators and site operators

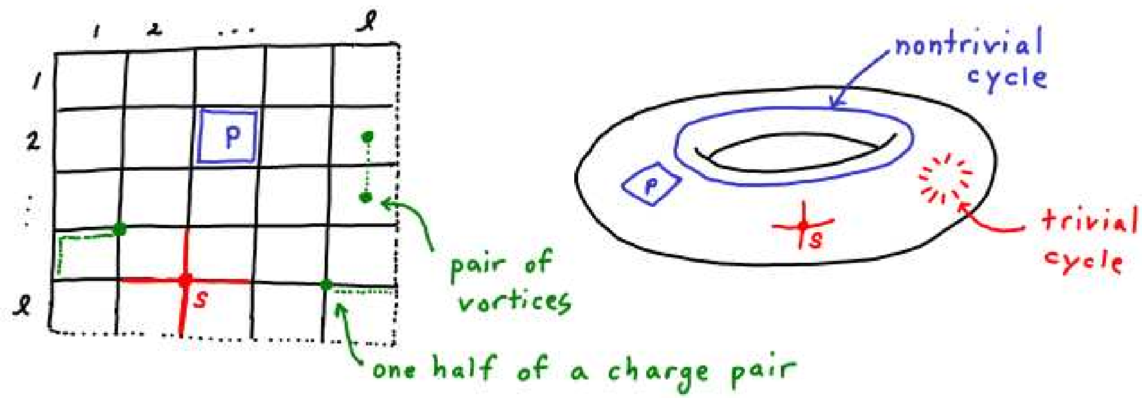
$$A_p := \bigotimes_{e \in \text{boundary}(p)} Z_e \text{ and } B_s := \bigotimes_{e \in \text{star}(s)} X_e \tag{1}$$

where e denotes an edge, star(s) denotes the set of edges that connect to a site s, and boundary(p) denotes the set of edges that bound a plaquette p. A measurement detects a magnetic defect (or vortex) on plaquette p if -1 is obtained on measuring A_p and an electric defect (or charge) on site s if -1 is obtained on measuring B_s. The defect-free vacuum can be represented as the ground state of the Hamiltonian

$$H = -\sum_{\text{plaquettes}} A_p - \sum_{\text{sites}} B_s. \tag{2}$$

Elementary excitations on the torus are charges with energy 2 and vortices with energy 2. If the system is excited thermally, pairs of charges and vortices may appear on the torus. We might expect that these pairs will tend to annihilate quickly due to some relaxation process if the temperature is significantly less than the gap between the ground state and first excited state. The purpose of this problem is to explore some basic properties of this Hamiltonian and its ground state.

(a) Show that the ground state is 4-fold degenerate. By cooling the system down, we expect that the system will relax into a state in this manifold. In fact, it is possible in principle to prepare a known state.



(b) Recall that a nontrivial cycle of the torus is a closed loop on the surface of the torus that cannot be deformed smoothly to a point, whereas a trivial cycle can be deformed to a point. Show that rotations occur in the ground state manifold when the following happens: a defect pair is created, one half of the pair traverses a nontrivial cycle of the torus, and the pair is destroyed. Show that a charge traversing a nontrivial cycle in this way produces a rotation that anticommutes with the rotation produced by a vortex traversing a different nontrivial cycle.

(c) Suppose that a weak field is applied locally to each qubit so that

$$H' = H + \epsilon \sum_{\text{edges}} C_e \tag{3}$$

where ϵ is a small real constant and C_e is a single qubit observable applied to the qubit on edge e. Show that the energy splitting between any two orthogonal ground states only appears in the ℓth order of perturbation theory and higher. This suggests that the ground state is robust to local perturbations if ϵ is small enough and ℓ is large enough.

The Hamiltonian H may be difficult to engineer because it has four-body terms. However, it is possible to repeatedly project the system into one of the eigenstates of H using a quantum circuit with only nearest-neighbor two-qubit gates, Pauli preparations and measurements, and ancilla. Once the energy eigenstate is identified, it is possible to determine a corrective operation that returns the system to the ground state of H, provided that the temperature is low enough.

(d) Place an additional ancilla qubit on each plaquette and each site. The ancilla qubits can begin in whatever state is convenient. Give a circuit that uses each ancilla qubit to measure the corresponding plaquette or site operator using only the set of operations that has just been described.

P2: (**Accuracy threshold for quantum error-correction**) The smallest error-correcting Bacon-Shor code is defined by the stabilizer generators,

$$g_1 = \texttt{XXXXXXIII},\ g_2 = \texttt{IIIXXXXXX},\ g_3 = \texttt{ZZIZZIZZI},\ g_4 = \texttt{IZZIZZIZZ}, \tag{4}$$

the normalizer operators for four "gauge" qubits,

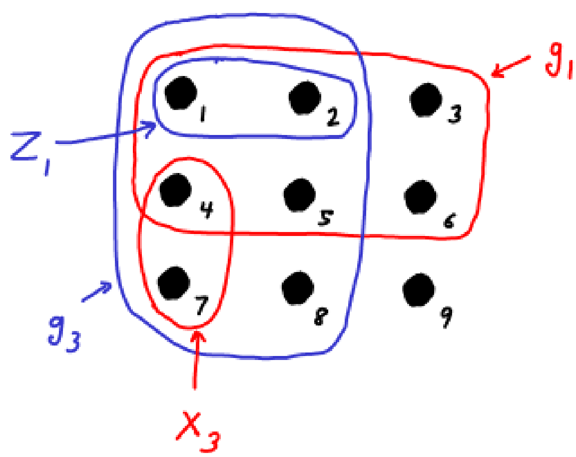
$$Z_1 = \texttt{ZZIIIIIII} \text{ and } X_1 = \texttt{XIIIIIXII},\ Z_2 = \texttt{IZZIIIIII} \text{ and } X_2 = \texttt{IIXIIIIIX}, \tag{5}$$

$$Z_3 = \texttt{IIIZZIIII} \text{ and } X_3 = \texttt{IIIXIIXII},\ Z_4 = \texttt{IIIIZZIII} \text{ and } X_4 = \texttt{IIIIIXIIX} \tag{6}$$

and the normalizer operators for one encoded logical qubit,

$$Z_5 = \texttt{ZZZZZZZZZ} \text{ and } X_5 = \texttt{XXXXXXXXX}. \tag{7}$$

It is convenient to view these on a 3×3 grid of qubits.



When logical qubits 1 through 4 are initialized in the state $|\bar{0}\bar{0}\bar{0}\bar{0}\rangle$, this code is exactly Shor's $[[9,1,3]]$ code. The Bacon-Shor construction enlarges the original code space to include a "gauge subsystem". The Hilbert space of all 9 qubits decomposes as $\mathcal{H} = \bigoplus(\mathcal{H}_L \otimes \mathcal{H}_S)$ where the direct sum is over the eigenspaces of the stabilizer, $\mathcal{H}_L$ is the logical qubit, and $\mathcal{H}_S$ is the gauge subsystem. The purpose of this problem is to understand that the error-correction circuitry can be simplified for some subsystem codes, and simplicity is usually good for fault-tolerance.

(a) Assuming that errors on the gauge qubits can be neglected, give the set of one and two-qubit errors which can be corrected (or ignored).

(b) It is acceptable to put the gauge qubits into any convenient state or even measure them. Explain how the syndrome can be determined by measuring the gauge qubits directly, and explain how to perform error correction of the encoded qubit based on the syndrome.

(c) Give fault-tolerant circuits for error-correction based on measuring gauge qubits. Begin by showing that X_1 can be measured fault-tolerantly without using a cat state as long as the circuit is repeated some number of times. Next, show that each of the stabilizer generators can be measured fault-tolerantly by a suitable combination of these kinds of circuits.

(d) Give a simple lower bound on the accuracy threshold for quantum error-correction based on your circuit. In this case, and provided your circuits are fault-tolerant in the sense described in class, a rigorous bound can be obtained for stochastic noise by counting the locations in *two* error-correction circuits, the circuit under consideration and the circuit leading into it (this object is called an extended rectangle).

(e) (Extra-credit +3) Show how to obtain this code from a toric code using only Pauli measurements and no ancilla.

P3: (**Codeword stabilized codes**) The purpose of this problem is to understand graph states and how they can be used to construct quantum error-correcting codes. A codeword stabilized (CWS) code is a quantum code that is spanned by a set of stabilizer states $W|S\rangle$ for $W \in \mathcal{W}$ where $|S\rangle$ is a state stabilized by an abelian subgroup S of the n-qubit Pauli group and $\mathcal{W}$ is a set of n-qubit Pauli operators such that $W_i|S\rangle \neq W_j|S\rangle$ for distinct elements $W_i, W_j \in \mathcal{W}$. Such a code will encode a $|\mathcal{W}|$-dimensional system into the Hilbert space of n qubits.

Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. A graph state $|G\rangle$ associated with G is the stabilizer state whose stabilizer is generated by $\langle \forall v \in V, X_v \otimes \bigotimes_{w \in N(v)} Z_w \rangle$ where $N(v)$ denotes the set of vertices that are adjacent to a vertex v. If the generators are written as a matrix, the matrix has X operators on the diagonal and Z operators in every position where there is a 1 in the adjacency matrix of G.

(a) Prove that any stabilizer state can be transformed into a graph state by applying a single qubit Clifford gate to each qubit.

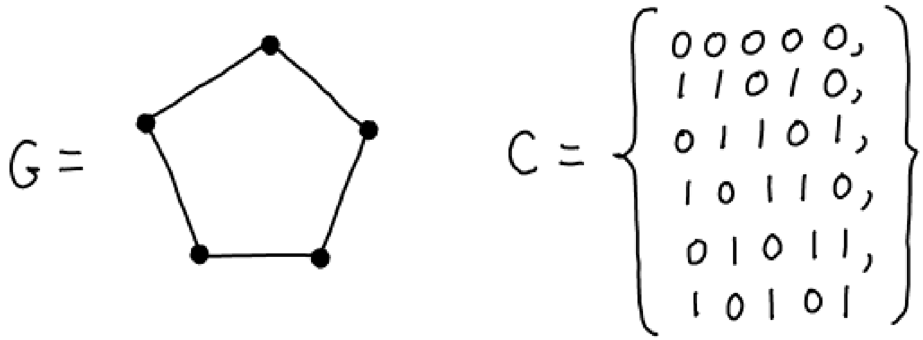
By applying single qubit Clifford gates, every CWS code can be put into a form where the stabilizer state $|S\rangle$ becomes a graph state $|G\rangle$ and the codeword operators W_i each become a tensor product of Zs and Is. The codeword operators can be written as $Z^{\mathbf{c}} = Z^{c_1} Z^{c_2} \ldots Z^{c_n}$ for an n-bit binary vector $\mathbf{c}$ in some classical code C. Therefore, a CWS code is specified using a graph G and a classical code C.

The error-correction conditions for CWS codes become simple and classical-like when the CWS code is put into a standard form. The graph defines a map from the n-qubit Pauli group to the set of n-bit strings

$$\text{classical}_G(E = \pm Z^{\mathbf{v}} X^{\mathbf{u}}) = \mathbf{v} \oplus \bigoplus_{l=1}^{n} (\mathbf{u})_l \mathbf{r}_l \tag{8}$$

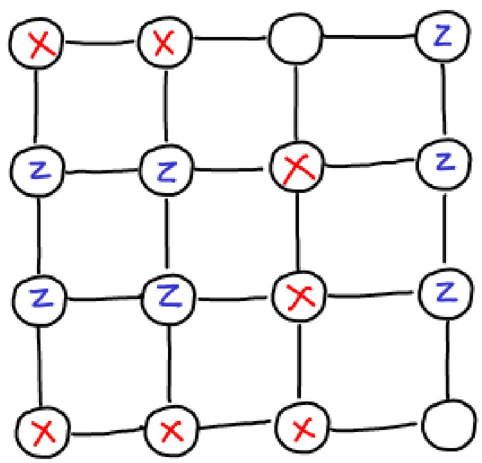
where $\mathbf{v}$ and $\mathbf{u}$ are n-bit strings that give the locations of Z and X terms in E (respectively), $(\mathbf{u})_l$ is the lth bit of $\mathbf{u}$, and $\mathbf{r}_l$ is the lth row of the adjacency matrix of G. This map essentially says "to translate a Pauli error into a classical error, place a 1 on every vertex where there is an Z or Y error, then, wherever there is an X or Y error, place a 1 on every *neighbor* of that vertex." A CWS code in standard form with graph G and codeword operators $\{Z^{\mathbf{c}}\}_{\mathbf{c} \in C}$ detects Pauli errors from $\mathcal{E}$ iff C detects errors from $\text{classical}_G(\mathcal{E})$ and in addition we have for each E, $\text{classical}_G(E) \neq 0$ or $\forall i\ Z^{\mathbf{c}_i} E = E Z^{\mathbf{c}_i}$.

(b) Show that the $((5, 6, 2))$ CWS code defined in the following figure can detect any single qubit error but cannot detect some two qubit errors.



(c) Express the $[[7, 1, 3]]$ Steane code as a CWS code in standard form by giving the graph G and the classical code C. Show that your CWS code detects all one and two qubit errors.

P4: (**Measurement-based computing with cluster states**) The purpose of this problem is to work through an example of a brief measurement-based computation. In the figure below, each measurement axis is written on the corresponding qubit.



(a) Which qubits are the input qubits and which are the output qubits?

(b) What is the state of the unmeasured qubits after the appropriate Pauli corrections?

(c) What quantum circuit has been simulated?