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Quantum Information Science

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Problem Set #2 Solutions

Problems:

P1: (**Measurements and stabilizers**) **(a)** First let us establish some notation. Suppose g is an n-qubit operator that is both Hermitian and unitary so that $g = g^{-1}$. The operator $P(g) := \frac{1}{2}(I + g)$ is a projector onto the $+1$ eigenspace of g. Indeed, $P(g)^2 = \frac{1}{4}(I + 2g + g^2) = P(g)$ and $gP(g) = P(g)$. The measurement of an n-qubit Pauli operator M has elements $P_k := P((-1)^k M)$. The stabilizer state $|\psi\rangle$ stabilized by $S = \langle g_1, g_2, \ldots, g_n\rangle$ so

$$\rho = |\psi\rangle\langle\psi| = \prod_{i=1}^{n} P(g_i). \tag{1}$$

The probability of outcome $k \in \{0, 1\}$ is

$$p_k := \mathrm{tr}(P_k\rho) = \frac{1}{2} + (-1)^k \frac{1}{2^{n+1}} \mathrm{tr}\left[\prod_{i=1}^{n}(M + Mg_i)\right]. \tag{2}$$

The trace is zero if $\pm M \notin S$ and otherwise it is $2^n(-1)^\ell$ if $(-1)^\ell M \in S$ so

$$p_k = \begin{cases} \frac{1}{2} \text{ if } \pm M \notin S \\ \frac{1}{2} + (-1)^{k+\ell}\frac{1}{2} \text{ if } (-1)^\ell M \in S. \end{cases} \tag{3}$$

Recall that $\pm M \notin S$ iff M anticommutes with some element of S. The post-measurement state when outcome k is obtained is (provided that $p_k \neq 0$)

$$\rho_k := p_k^{-1} P_k \rho P_k = \frac{p_k^{-1}}{2^{n+2}}\left[\Sigma + (-1)^k\{M, \Sigma\} + M\Sigma M\right] \tag{4}$$

where $\Sigma := \sum_{g\in S} g$. If $(-1)^k M \in S$ for some $k \in \{0, 1\}$ then $P_k\rho = \rho$ and $p_k = 1$, so $\rho_k = \rho$. Otherwise, if $\pm M \notin S$ then M anticommutes with some generators $g_1, g_2, \ldots, g_j$, $1 \le j \le n$, and $p_k = \frac{1}{2}$ for $k \in \{0, 1\}$. Let S_0 denote the subgroup of S that commutes with M. We can form new generators $g_1g_2, g_1g_3, \ldots, g_1g_j$ that commute with M so that S_0 is generated by these new generators together with $g_{j+1}, g_{j+2}, \ldots, g_n$. Therefore, $|S_0| = |S|/2$. The expression in square brackets in (4) becomes

$$2(I + (-1)^k M)\sum_{g\in S_0} g. \tag{5}$$

Therefore, ρ_k is the pure state stabilized by $S' := S_0 \cup (-1)^k M S_0$.

(b) Since IY anticommutes with both XX and ZZ, the post-measurement state is stabilized by $\langle(-1)^k IY, -YY\rangle$ where the measurement outcome k is 0 or 1, each with probability 1/2.

(c),(d) Initially, and leaving the phase factors $\{\pm 1, \pm i\}$ out of the generating sets of the normalizers,

$$S = \langle IZ\rangle \text{ and } N(S)/S = \langle XI, ZI\rangle. \tag{6}$$

Let u denote the outcome of measuring YX and v denote the outcome of measuring IY. After measuring YX,

$$S_1 = \langle(-1)^u YX\rangle \text{ and } N(S_1)/S_1 = \langle XZ, ZZ\rangle. \tag{7}$$

We have applied the rule that the normalizer elements are multiplied by the anticommuting element of the stabilizer if they also anticommute with the measured operator. Now we measure IY and obtain

$$S_2 = \langle(-1)^v IY\rangle \text{ and } N(S_2)/S_2 = \langle(-1)^{u+1} ZY, (-1)^u XY\rangle. \tag{8}$$

We are free to choose new coset representatives $N(S_2)/S_2 = \langle(-1)^{u+v+1} ZI, (-1)^{u+v} XI\rangle$. Therefore, this sequence of measurements applies a Hadamard gate up to some element of the Pauli group and leaves the bottom qubit in an eigenstate of Y. The signs can be removed from the coset representatives by applying Z to the first qubit if $u+v$ has odd parity and X if $u+v$ has even parity.

(e) The initial stabilizer is now $S = \langle IX\rangle$ so $S_1 = \langle IX, (-1)^u YI\rangle$ and $S_2 = \langle(-1)^v IY, (-1)^u YI\rangle$. The outcome of the first measurement depends on the state of the top qubit. The normalizer is trivial. After applying the corrections from parts (c,d), $S_2 = \langle(-1)^v IY, (-1)^{u+1} YI\rangle$. The circuit is pretty worthless unless you want a random eigenstate of Y.

P2: (**Surface codes**) **(a)** There are four distinct types of generators: plaquette operators with weight 3 and 4 and site operators with weight 3 and 4. They are clearly independent. The site operators commute with each other and the plaquette operators commute with each other. A plaquette operator and site operator are either disjoint or overlap on two coordinates. Therefore all of the generators commute. There are $\ell(\ell-1)$ plaquette operators and $2(\ell-1)$ of these have weight 3. Similarly, there are also $\ell(\ell-1)$ site operators. Therefore, the code encodes $k = \ell^2 + (\ell-1)^2 - 2\ell(\ell-1) = 1$ qubit, so it is a $[[n = \ell^2 + (\ell-1)^2, k = 1]]$ stabilizer code. In fact, it is a CSS code.

(b) $N(S)$ coincides with the centralizer of S, i.e. it contains all the Pauli operators that commute with S. We will neglect a phase when giving generating sets for $N(S)$. Therefore, $N(S)$ is generated by the generating set of S together with $2k = 2$ additional operators. Number the qubits on $\mathcal{L}$ from left to right and top to bottom starting with the vertical edges first and continuing with the horizontal links. In other words, the top left qubit is qubit 1 and the qubit on the vertical edge directly below it is $\ell+1$. The qubit that shares a site operator with qubit 1 and $\ell+1$ is labeled by ℓ^2+1. By inspection, the operators

$$\bar{X} := X_1 X_2 X_3 \dots X_\ell \text{ and } \bar{Z} := Z_1 Z_{\ell+1} Z_{2\ell+1} \dots Z_{(\ell-1)\ell+1} \tag{9}$$

commute with S and are independent of S and one another. They anticommute as required. Therefore, $N(S)/S = \langle\bar{X}, \bar{Z}\rangle$. By inspection, $g\bar{X}$ and $g\bar{Z}$ will have weight greater than or equal to ℓ for any $g \in S$, so $\bar{X}$ is a minimum weight element in $N(S) - S$ with weight ℓ. Therefore, the code has parameters $[[n = \ell^2 + (\ell-1)^2, k = 1, d = \ell]]$.

(c) Neglecting phase factors again, the normalizer mod S equals

$$N(S)/S = \{S, \bar{X}S, \bar{Z}S, \bar{Z}S \cdot \bar{X}S\}. \tag{10}$$

The coset S acts like the identity. It consists Z operators that form closed loops in the bulk or that form closed loops with respect to one of the "rough" north or south boundaries. It also consists of X operators that form closed loops or that form closed loops with respect to one of the "smooth" west or east boundaries. Finally, it consists of products of these close loops of mixed X and Z type. Therefore, $\bar{X}S$ consists of X operators that "begin" on the east or west smooth boundary and "end" on the opposite smooth boundary, since they can only deviate from $\bar{X}$ by closed loops of X and Z operators. Similarly, $\bar{Z}S$ begins on the north or south rough boundary and ends on the opposite boundary.

(d) Let us remove the plaquette associated with qubits $\ell+1$, $\ell+2$, ℓ^2+1, and $\ell^2+\ell$. The stabilizer is now generated by one fewer element, so the new code must have parameters $[[n, k=2]]$. The normalizer must be generated by two new elements, one of which is A_{p_0}. Indeed, $X_{\ell+1}$ now commutes with the stabilizer, so it can be chosen as a new generator of the normalizer. Since it has weight 1, the new code is $[[n, k=2, d=1]]$. This example isn't particularly interesting because the distance is terrible. However, if you imagine that removing the plaquette has "created a smooth hole" in the surface, then you would observe that the new Z-type normalizer elements "encircle" the hole and the X-type normalizer elements begin on the hole and end on a smooth boundary. You might imagine that each hole encodes an additional qubit and that the perimeter and separation of the holes determines the new distance.

P3: (**Logic gates for a** 15 **qubit code**) **(a)** Every generator has even weight. Therefore we can choose $\bar{X} = X^{\otimes 15}$ and $\bar{Z} = Z^{\otimes 15}$ as generators of $N(S)/\{e^{i\phi}S\}$. These operators anticommute as expected, and since there are 14 generators, we have a complete set of generators of the normalizer. The product of $\bar{Z}$, $I^{\otimes 7}Z^{\otimes 8}$, $I^{\otimes 3}Z^{\otimes 4}I^{\otimes 4}Z^{\otimes 4}$, and $I^{\otimes 11}Z^{\otimes 4}$ is in $N(S) - S$, has weight 3, and there is no operator in $N(S) - S$ of lower weight. Therefore, this code is a $[[15, 1, 3]]$ code.

(b) The CNOT gate "propagates" X operators from control to target and Z operators from target to control. Therefore, since each generator is a product of a single type of Pauli operator, **CNOT** maps $S \times S$ to $S \times S$ by conjugation. Furthermore, $\bar{X}$ propagates from control to target of **CNOT** and $\bar{Z}$ propagates from target to control. This is sufficient to show that **CNOT** is a logical CNOT gate.

(c) **H** exchanges $\bar{X}$ and $\bar{Z}$. However, the Z-type generators are not mapped into S under conjugation by $H^{\otimes 15}$. Therefore, $\mathbf{H}P_Q\mathbf{H}^\dagger \neq P_Q$ where P_Q is the projector onto the $[[15, 1, 3]]$ code, meaning that the code space is not preserved by **H**.

(d) We know that the $[[15, 1, 3]]$ code is the span of vectors $|\bar{0}\rangle = \frac{1}{\sqrt{|C_2^\perp|}} \sum_{c \in C_2^\perp} |c\rangle$ and $|\bar{1}\rangle = \bar{X}|\bar{0}\rangle$. We can directly compute

$$\mathbf{T}^\dagger|\bar{0}\rangle = \frac{1}{\sqrt{|C_2^\perp|}} \sum_{c \in C_2^\perp} e^{i\,\mathrm{wt}(c)\pi/4}|c\rangle = |\bar{0}\rangle \tag{11}$$

since the codewords of $C_2^\perp$ have weight 0 or weight 8, and

$$\mathbf{T}^\dagger|\bar{1}\rangle = e^{i\{7 \text{ or } 15\}2\pi/8}|\bar{1}\rangle = e^{-i\pi/4}|\bar{1}\rangle. \tag{12}$$

How does $\mathbf{T}^\dagger$ act by conjugation on S? It must be that $\bar{T}^\dagger S \bar{T} \subseteq \mathcal{S}$ where $\mathcal{S}$ is a generalized stabilizer $\{U \in SU(2)^{\otimes n} \mid U|\psi\rangle = |\psi\rangle\ \forall |\psi\rangle \in C(S)\}$ that contains S. When does $S = \mathcal{S}$?