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Quantum Information Science

February 14, 2008

Problem Set #2

(due in class, 28-Feb-08)

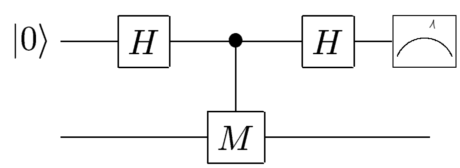
Lecture Topics (2/14, 2/21, 2/26): Stabilizer codes; exotic codes; computing on codes

Recommended Reading: Nielsen and Chuang, Section 10.5

Problems:

P1: (**Measurements and stabilizers**) Stabilizers are one of the most useful way to describe states and transformations in quantum information. In this problem we investigate how measurement is described using the stabilizer formalism. Recall that if an n-qubit state $|\psi\rangle$ is stabilized by $S = \langle g_1, g_2, \ldots, g_n\rangle$, then $g|\psi\rangle = |\psi\rangle$ for all $g \in S$. Note that $g_1, g_2, \ldots, g_n$ are the *generators* of S.

(a) Let M be an n qubit measurement operator (expressed as a product of pauli operators, as usual in the stabilizer formalism). Recall that we use the circuit



to measure M on $|\psi\rangle$, and this gives measurement outcomes $k = \{0, 1\}$ (corresponding to the $+1$ and -1 eigenstates, respectively). Show that if M commutes with g_j for all j, then the post-measurement state is stabilized by S. But if M anticommutes with some of the generators, say $g_1, g_2, \ldots, g_j$ (and commutes with $g_{j+1}, \ldots, g_n$), then the post-measurement state is stabilized by

$$\langle (-1)^k M, g_1 g_2, g_1 g_3, \ldots, g_1 g_j, \ldots, g_{j+1}, g_{j+2}, \ldots, g_n \rangle \,. \tag{1}$$

(b) Let $|\psi\rangle$ be stabilized by $\langle XX, ZZ\rangle$. What is the stabilizer of the state after measuring IY?

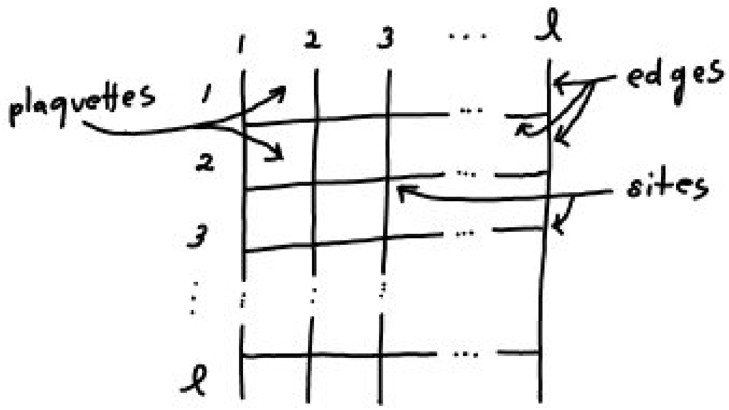
(c) Let $|\psi\rangle$ be a single qubit state, and suppose we start with a system in the state $|\psi\rangle \otimes |0\rangle$, measure $Y \otimes X$, and then measure $I \otimes Y$. What Pauli operations do we need to perform following each of the measurements to steer the post-measurement state into the $+1$ eigenstate of each measured operator?

(d) Compute the action of the above series of operations (with Pauli corrections) on the $\bar{X}$ and $\bar{Z}$ operators (the normalizer for the initial state $|0, \psi\rangle$). Describe the overall action in terms of standard gates.

(e) Suppose we had started with the input state $|\psi\rangle \otimes (|0\rangle + |1\rangle)/\sqrt{2}$ instead, and performed the same two measurements. What would have happened then?

P2: (**Surface codes**) This problem defines a family of stabilizer codes whose stabilizers have low weight generators determined by a lattice. These codes are interesting for many reasons including their excellent fault-tolerance properties and their relationship to statistical mechanics models.

Consider the $\ell \times \ell$ lattice $\mathcal{L}$ shown below and let $\ell > 1$.



The lattice consists of a collection of plaquettes p, sites s, and edges e. Place a qubit on each edge of the lattice and consider the stabilizer S generated by *plaquette operators* $A_p = \otimes_{e \in N(p)} Z_e$ associated with each plaquette and *site operators* $B_s = \otimes_{e \in N(s)} X_e$ associated with each site. $N(p)$ denotes the set of edges neighboring p, which are the four edges on the perimeter of p. Likewise, $N(s)$ denotes the set of edges neighboring s, which are the four edges connected to s.

(a) Show that $S := \langle A_p, B_s, \forall p, s \in \mathcal{L} \rangle$ is a valid stabilizer and give the dimension of the associated stabilizer code $C(S)$.

(b) Give the generators for the normalizer $N(S)$ of the stabilizer in the Pauli group. How do you know you have generated all of $N(S)$ (in the Pauli group)? What is the distance of $C(S)$?

(c) Two elements of the same coset in $N(S)/S$ may act on disjoint sets of qubits, but they both share a common boundary or both have no boundary. Explain how the members of each coset in $N(S)/S$ are topologically related in terms of the north, south, east, and west boundaries of the lattice.

(d) Suppose a single plaquette operator A_{p_0} along the west boundary of the lattice is removed from S to give a new stabilizer $S' := \langle A_p, B_s, \forall p \neq p_0, s \in \mathcal{L} \rangle$. What is the dimension and minimum distance of this new code and why?

P3: (**Logic gates for a** 15 **qubit code**) In this problem, we will study some logic gates that can be performed bitwise on a stabilizer code. Consider the code $C(S)$ whose stabilizer S is generated by

$$\begin{bmatrix} I & I & I & I & I & I & I & Z & Z & Z & Z & Z & Z & Z & Z \\ I & I & I & Z & Z & Z & Z & I & I & I & I & Z & Z & Z & Z \\ I & Z & Z & I & I & Z & Z & I & I & Z & Z & I & I & Z & Z \\ Z & I & Z & I & Z & I & Z & I & Z & I & Z & I & Z & I & Z \\ I & I & I & I & I & I & I & X & X & X & X & X & X & X & X \\ I & I & I & X & X & X & X & I & I & I & I & X & X & X & X \\ I & X & X & I & I & X & X & I & I & X & X & I & I & X & X \\ X & I & X & I & X & I & X & I & X & I & X & I & X & I & X \\ I & I & I & I & I & I & I & I & I & I & I & Z & Z & Z & Z \\ I & I & I & I & I & I & I & I & I & Z & Z & I & I & Z & Z \\ I & I & I & I & I & I & I & I & Z & I & Z & I & Z & I & Z \\ I & I & I & I & I & Z & Z & I & I & I & I & I & I & Z & Z \\ I & I & I & I & Z & I & Z & I & I & I & I & I & Z & I & Z \\ I & I & Z & I & I & I & Z & I & I & I & Z & I & I & I & Z \end{bmatrix}. \tag{2}$$

(a) What are the parameters for this code? What are the Pauli $\bar{X}$ and $\bar{Z}$ operators on the encoded qubit(s)?

(b) Using the stabilizer formalism, show that the transversal CNOT gate $\mathbf{CNOT} := CNOT^{\otimes 15}$ is a logic gate for this code.

(c) Show that the bitwise Hadamard gate $\mathbf{H} := H^{\otimes 15}$ is *not* a logic gate for this code.

(d) This code is a CSS(C_1, C_2) code constructed from a classical first order punctured Reed-Muller code $C_1 = RM^*(1,4)$ and its even subcode $C_2^\perp = \text{even}(RM^*(1,4)) \subseteq C_1$. The dual code of both $C_2^\perp$ and C_1 is a classical second order Reed-Muller code $RM^*(2,4)$. The generator matrix for $C_2^\perp$ is

$$H_{C_2^\perp} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \tag{3}$$

and C_1 is generated by the same 4 vectors as $C_2^\perp$ as well as the all ones vector 111111111111111. The dual code $C_2 = C_1^\perp$ is generated by the matrix of Z-type stabilizer generators where Z has been replaced by 1 and I by 0. By operating on the state vectors, show that the bitwise $T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/8} \end{pmatrix}$ gate $\mathbf{T}^\dagger := T^{\otimes 15}$ is a logic gate for this code that applies a $T^\dagger$ gate on the encoded qubit(s).