MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Department of Physics, EECS, and Department of Applied Math

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Quantum Information Science

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Problem Set #1 Solutions

Problems:

P1: (**Quantum operations**) (a) This channel is an amplitude damping channel in the limit of strong damping, $\mathcal{E} = \{|0\rangle\langle 0|, |0\rangle\langle 1|\}$,

$$\mathcal{E}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} d & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a+d & 0 \\ 0 & 0 \end{bmatrix} = |0\rangle\langle 0| \tag{1}$$

This can be obtained without relying on an amplitude damping channel by observing the last equality $|0\rangle\langle 0| = \begin{bmatrix} a+d & 0 \\ 0 & 0 \end{bmatrix}$.

(b) This channel is a completely depolarizing channel, $\mathcal{E} = \{\frac{1}{2}X, \frac{1}{2}Y, \frac{1}{2}Z, \frac{1}{2}I\}$.

$$\mathcal{E}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \frac{1}{4}\left(\begin{bmatrix} d & c \\ b & a \end{bmatrix} + \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} + \begin{bmatrix} a & -b \\ -c & d \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \tag{2}$$

$$= \frac{1}{2}\begin{bmatrix} a+d & 0 \\ 0 & a+d \end{bmatrix} = I/2 \tag{3}$$

One way to derive this is as follows. We are looking for a quantum operation that maps the Bloch sphere onto its center. No more than d^2 operation elements are necessary. If we could rotate a state by π about the appropriate axis, the sum of the original state and the rotated state would be the state in the center of the sphere. However, since *any* point must be mapped to $I/2$, there can be no preferred axis of rotation, so we can first consider rotation about, say, the X axis by π. The (unnormalized) result $\rho + X\rho X$ lies on the X axis, so rotating *this* state about the Z axis by π gives the desired output.

(c) This channel completely depolarizes with probability p and can be obtained from the quantum operation $\mathcal{E}_b$ from part (b),

$$\mathcal{E}(\rho) = pI/2 + (1-p)\rho = p\mathcal{E}_b(\rho) + (1-p)\rho \tag{4}$$

$$= \frac{p}{4}\left[X\rho X + Y\rho Y + Z\rho Z + \rho\right] + (1-p)\rho \tag{5}$$

$$= (1 - \frac{3}{4}p)\rho + \frac{p}{4}\left[X\rho X + Y\rho Y + Z\rho Z\right], \tag{6}$$

so $\mathcal{E} = \{\sqrt{1-\frac{3}{4}p}I, \sqrt{\frac{p}{4}}X, \sqrt{\frac{p}{4}}Y, \sqrt{\frac{p}{4}}Z\}$.

(d) The rotation is

$$R_x(\theta) = e^{-i\theta X/2} = \begin{bmatrix} \cos\theta/2 & -i\sin\theta/2 \\ -i\sin\theta/2 & \cos\theta/2 \end{bmatrix} \tag{7}$$

so the operation elements that arise from tracing out the environment are

$$E_0 = \langle 0_e|\Lambda(R_x(\theta))|0_e\rangle = \begin{bmatrix} 1 & 0 \\ 0 & \cos\theta/2 \end{bmatrix} \tag{8}$$

$$E_1 = \langle 1_e|\Lambda(R_x(\theta))|0_e\rangle = \begin{bmatrix} 0 & 0 \\ 0 & -i\sin\theta/2 \end{bmatrix}. \tag{9}$$

We can write $\cos\theta/2 = e^{-\lambda}$ and $\sin\theta/2 = \sqrt{1-e^{-2\lambda}}$.

(e) By inspection, $\rho_{out} = \alpha\rho_{in} + (1-\alpha)Z\rho_{in}Z$, so $E_0 = \sqrt{\alpha}I$ and $E_1 = \sqrt{1-\alpha}Z$. For an input state

$$\rho_{in} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{10}$$

the output state for part (d) is

$$\rho_{out,d} = \begin{bmatrix} a & e^{-\lambda}b \\ e^{-\lambda}c & d \end{bmatrix} \tag{11}$$

and the output state for part (e) is

$$\rho_{out,e} = \begin{bmatrix} a & (2\alpha-1)b \\ (2\alpha-1)c & d \end{bmatrix}. \tag{12}$$

Therefore, the boxes in parts (d) and (e) are represented by the same quantum operation and $e^{-\lambda} = 2\alpha - 1$.

P2: (**A 5 qubit code**) The solution for the original projector is given first. The projector for this problem is

$$P := \frac{1}{16}\left[3I + IZYYZ_c + IXZZX_c - IYXXY_c + 2ZXYYX - 2ZZZZZ\right], \tag{13}$$

where, for example, $IZYYZ_c$ denotes a sum of cyclic shifts $(IZYYZ + ZIZYY + YZIZY + YYZIZ + ZYYZI)$. Consider the operator

$$\Sigma := I + IZYYZ_c + IYXXY_c + IXZZX_c - ZIXXI_c + ZXYYX_c - ZYIIY_c + ZZZZZ. \tag{14}$$

The terms of Σ form a group so $g\Sigma = \Sigma$ for any g appearing in Σ. Rewrite the projector in terms of Σ,

$$16P = 2I + \Sigma - 2IYXXY_c + ZIXXI_c + ZXYYX_c + ZYIIY_c - 3ZZZZZ. \tag{15}$$

All of the terms commute.

(a) We must first show that $P^2 = P$. There are 28 terms in the product P^2. The following identities are not too hard to verify:

$$ZIXXI_c^2 = 5I + 2ZZXIX_c + 2YIYXX_c \tag{16}$$
$$ZXYYX_c^2 = 5I + 2YZIZY_c + 2XZZXI_c \tag{17}$$
$$ZYIIY_c^2 = 5I + 2XXYIY_c + 2ZIZYY_c \tag{18}$$
$$IYXXY_c^2 = 5I + 2YYZIZ_c + 2XIXZZ_c \tag{19}$$
$$g_c\Sigma = -5\Sigma \text{ if } -g_c \in \Sigma \tag{20}$$
$$g_c\Sigma = 5\Sigma \text{ if } g_c \in \Sigma \tag{21}$$
$$ZIXXI_cZXYYX_c = -IXZZX_c - 2ZYYZI_c - 2YXXYI_c \tag{22}$$
$$ZIXXI_cZYIIY_c = IYXXY_c + 2ZXIXZ_c + 2YYZIZ_c \tag{23}$$
$$ZXYYX_cZYIIY_c = -IZYYZ_c - 2XYIYX_c - 2ZZXIX_c \tag{24}$$
$$IYXXY_cZIXXI_c = ZYIIY_c + 2IXXIZ_c - 2XYYXZ_c \tag{25}$$
$$IYXXY_cZXYYX_c = 5ZZZZZ - 2XXIZI_c - 2YZYII_c \tag{26}$$
$$IYXXY_cZYIIY_c = ZIXXI_c + 2IIYZY_c - 2YXZXY_c \tag{27}$$
$$\Sigma^2 = 32\Sigma. \tag{28}$$

Using the identities, we can compute the product

$$(16P)^2 = 13I + 4\Sigma + \Sigma^2 + ZIXXI_c^2 + ZXYYX_c^2 + ZYIIY_c^2 + 4ZIXXI_c + 4ZXYYX_c + 4ZYIIY_c \tag{29}$$
$$- 12ZZZZZ + 2ZIXXI_c\Sigma + 2ZXYYX_c\Sigma + 2ZYIIY_c\Sigma - 6ZZZZZ\Sigma + 2ZIXXI_cZXYYX_c \tag{30}$$
$$+ 2ZIXXI_cZYIIY_c - 6ZIXXI_cZZZZZ + 2ZXYYX_cZYIIY_c - 6ZXYYX_cZZZZZ \tag{31}$$
$$- 6ZYIIY_cZZZZZ - 8IYXXY_c - 4IYXXY_c\Sigma + 4IYXXY_c^2 - 4IYXXY_cZIXXI_c \tag{32}$$
$$- 4IYXXY_cZXYYX_c - 4IYXXY_cZYIIY_c + 12ZXYYX_c \tag{33}$$
$$= 48I + 16ZZXIX_c - 16YIYXX_c + 16YZIZY_c + \tag{34}$$
$$0ZIXXI_c + 32ZXYYX_c + 0ZYIIY_c - 32ZZZZZ \tag{35}$$
$$= 16^2P + 16\Sigma - 16\Sigma = 16^2P. \tag{36}$$

Therefore, P is a projection. We need to show that P is rank 6. Since $P^n = P$ for any integer $n \geq 1$, the eigenvalues of P must be either 0 or 1 and $\text{Rank}(P) = \text{Tr } P$. The trace of all the non-identity terms is zero since $\text{Tr}(A \otimes B) = \text{Tr}(A)\text{Tr}(B)$ and the Pauli matrices X, Y, and Z are traceless. Therefore, $\text{Tr } P = \frac{3}{16}\text{Tr}I = 6$.

(b) A sufficient condition for detecting a set of errors $\{E\}$ is that $PEP = 0$ for all E. To show that the code corrects all single qubit erasures, it is sufficient to check the first coordinate since the projector is cyclic. Therefore, we should show that $PX_1P = 0$, $PZ_1P = 0$, and $PY_1P = 0$. This is best done numerically since the errors destroy symmetries that make the calculation brief in part (a). The following Mathematica code does this.

```
X = {{0, 1}, {1, 0}}; Y = {{0, -i}, {i,0}};
Z = {{1, 0}, {0, -1}}; Id = IdentityMatrix[2];
```

```
KP[T_, U_, V_, W_, X_] :=
  KroneckerProduct[
   KroneckerProduct[
    KroneckerProduct[KroneckerProduct[T, U], V], W],
   X];
P := 1/16 (3*KP[Id, Id, Id, Id, Id] + KP[Id, Z, Y, Y, Z] +
     KP[Z, Id, Z, Y, Y] + KP[Y, Z, Id, Z, Y] + KP[Y, Y, Z, Id, Z] +
     KP[Z, Y, Y, Z, Id] + KP[Id, X, Z, Z, X] + KP[X, Id, X, Z, Z] +
     KP[Z, X, Id, X, Z] + KP[Z, Z, X, Id, X] +
     KP[X, Z, Z, X,
      Id] - (KP[Id, Y, X, X, Y] + KP[Y, Id, Y, X, X] +
       KP[X, Y, Id, Y, X] + KP[X, X, Y, Id, Y] +
       KP[Y, X, X, Y, Id]) +
     2*(KP[Z, X, Y, Y, X] + KP[X, Z, X, Y, Y] + KP[Y, X, Z, X, Y] +
        KP[Y, Y, X, Z, X] + KP[X, Y, Y, X, Z]) - 2*KP[Z, Z, Z, Z, Z]);
P.KP[X, Id, Id, Id, Id].P
P.KP[Y, Id, Id, Id, Id].P
P.KP[Z, Id, Id, Id, Id].P
```

(c) One can verify numerically that $PY_1X_2P \neq 0$, so $d \leq 2$. Together with the result from (b), this shows that $d = 2$. Here is the Mathematica code.

```
P.KP[Y, Id, Id, Id, Id].KP[Id, X, Id, Id, Id].P
```

The solution for the alternate version of this problem is given next.

(a) Π is the projector associated with a pure state whose stabilizer is $\langle XZIIZ_c \rangle$. The terms of Π form a group, so $\Pi^2 = \Pi$. Since every term except the identity is traceless, $\mathrm{Tr}\,\Pi = 1$.

(b) Let $\Pi(\mathbf{v})$ be the projector onto a pure state stabilized by $\langle (-1)^{v_1} g_1, (-1)^{v_2} g_2, \ldots, (-1)^{v_n} g_n \rangle$ where $\mathbf{v} = v_1 v_2 \ldots v_n$ is a binary string and $\{g_i\}$ is a set of n distinct pairwise commuting elements of the n-qubit Pauli group. Suppose $\mathbf{v} \neq \mathbf{w}$, then there is an index k for which $v_k \neq w_k$. Since

$$\Pi(\mathbf{v})\Pi(\mathbf{w}) = \frac{1}{2^{2n}} \prod_{i,j=1}^{n} (I + (-1)^{v_i} g_i)(I + (-1)^{w_j} g_j), \tag{37}$$

there is a term $(I + (-1)^{v_k} g_k + (-1)^{w_k} g_k + (-1)^{v_k + w_k} I)$ and therefore $\Pi(\mathbf{v})\Pi(\mathbf{w}) = 0$. Now, the projector Q can be written

$$Q = \Pi(00000) + \Pi(11010) + \Pi(01101) + \Pi(10110) + \Pi(01011) + \Pi(10101) \tag{38}$$

where the generators g_i are cyclic shifts of $XZIIZ$. Therefore, it is clear that $Q^2 = Q$ and $\mathrm{Tr}\,Q = 6$.

(c) If $QEQE^\dagger = 0$ and E is invertible, then $QEQ = 0$ and E is detectable. By inspection, each of the projectors

$$X_1QX_1 = \Pi(01001) + \Pi(10011) + \Pi(00100) + \Pi(11111) + \Pi(00010) + \Pi(11101) \tag{39}$$
$$Z_1QZ_1 = \Pi(10000) + \Pi(01010) + \Pi(11101) + \Pi(00110) + \Pi(11011) + \Pi(00101) \tag{40}$$
$$Y_1QY_1 = \Pi(11001) + \Pi(00011) + \Pi(10100) + \Pi(01111) + \Pi(10010) + \Pi(01101) \tag{41}$$

is orthogonal to Q. Since Q is invariant under cyclic shifts, Q detects all single qubit errors.

(d) The projector

$$Y_1X_2QX_2Y_1 = \Pi(01101) + \Pi(10111) + \Pi(00000) + \Pi(11011) + \Pi(00110) + \Pi(11000) \tag{42}$$

is not orthogonal to Q, so Q does not detect Y_1X_2. Therefore, $d = 2$.

P3: (**A 4 qubit CSS code**) (a) The dimension of the code is $k = 3$ so there are 8 codewords. Label the columns of G by $u = 1100$, $v = 1010$, and $w = 1111$. The code $\mathcal{C}$ is $\{u + u = 0000, u = 1100, u + w = 0011, v = 1010, v + w = 0101, u + v = 0110, u + v + w = 1001, w = 1111\}$. The procedure to column reduce G is

$$\left[\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ \hline u & v & w \end{array}\right] \mapsto \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ \hline u & u+v & u+w \end{array}\right] \mapsto \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ \hline u+v+w & v+w & u+w \end{array}\right] \tag{43}$$

so $A = [111]$.

(b) $H = [1111]$ by definition. Every codeword of $\mathcal{C}$ has even weight, so $Hx = 0$ for all $x \in \mathcal{C}$. $HG = \left[\begin{array}{cc} A & I \end{array}\right] \left[\begin{array}{c} I \\ A \end{array}\right] = 2A = 0$.

(c) For a binary linear code, $d(x, y) = d(x + y, y + y) = d(x + y, 0)$ for all $x, y \in \mathcal{C}$, so it suffices to identify the minimum weight over all nonzero codewords. Therefore, $d = 2$ for this example and the code can detect one error but cannot correct any errors.

(d) $\mathcal{C}^\perp$ is generated by $H^T = \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right]$ and $\mathcal{C}^\perp = \{0000, 1111\} \subset \mathcal{C}$ by inspection. We have $x \neq y \bmod \mathcal{C}^\perp \implies \langle\psi(y)|\psi(x)\rangle = |\mathcal{C}^\perp|^{-1} \sum_{u,v\in\mathcal{C}^\perp} \langle y + u|x + v\rangle = 0$ since $\langle y + u|x + v\rangle = 0$ for all u and v in the sum. The following states span the quantum code in this example:

$$|\psi(0000)\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle) \tag{44}$$
$$|\psi(1100)\rangle = \frac{1}{\sqrt{2}}(|1100\rangle + |0011\rangle) \tag{45}$$
$$|\psi(1010)\rangle = \frac{1}{\sqrt{2}}(|1010\rangle + |0101\rangle) \tag{46}$$
$$|\psi(0110)\rangle = \frac{1}{\sqrt{2}}(|0110\rangle + |1001\rangle) \tag{47}$$

$\mathcal{C}$ has distance 2, so any single bit error maps a codeword to a vector outside of $\mathcal{C}$. Therefore, any single qubit X error maps a basis state $|\psi(x)\rangle$ to an orthogonal vector outside the quantum code. Since each basis state has the same number of zeros and ones on each coordinate (i.e. a single zero and a single one), a single qubit Z error maps $|\psi(x)\rangle$ to an orthogonal vector outside the quantum code, eg:

$$\left(\frac{\langle 0000| + \langle 1111|}{\sqrt{2}}\right)\left(\frac{|0000\rangle - |1111\rangle}{\sqrt{2}}\right) = \frac{1}{2}(1-1) = 0. \tag{48}$$

If a single bit is flipped, the observation we used for Z errors still holds, so a single Y error maps $|\psi(x)\rangle$ to an orthogonal vector outside the quantum code as well. The code encodes $k - k^{\perp} = 3 - 1 = 2$ qubits.

(e) Generally for a CSS code of this form,

$$(C_{1,n+1}C_{2,n+2}\dots C_{n,2n})|\psi(x)\rangle \otimes |\psi(y)\rangle = |\mathcal{C}^{\perp}|^{-1} \sum_{u,v\in\mathcal{C}^{\perp}} |x+u\rangle|y+v+(x+u)\rangle \tag{49}$$

$$= |\mathcal{C}^{\perp}|^{-1} \sum_{u\in\mathcal{C}^{\perp}} |x+u\rangle \sum_{w\in\mathcal{C}^{\perp}} |x+y+w\rangle \tag{50}$$

$$= |\psi(x)\rangle \otimes |\psi(x+y)\rangle. \tag{51}$$

For this example, let $|\bar{0}\bar{0}\rangle := |\psi(0000)\rangle$, $|\bar{0}\bar{1}\rangle := |\psi(1100)\rangle$, $|\bar{1}\bar{0}\rangle := |\psi(1010)\rangle$, and $|\bar{1}\bar{1}\rangle := |\psi(0110)\rangle$, so it is clear that the gate $C_{15}C_{26}C_{37}C_{48}$ performs a CNOT between corresponding *encoded* qubits.