Quantum Information Science Notes

Andy Lutomirski

Spring 2007-08

Contents

1 Quantum games **1**
1.1 History . . . 1
1.2 The PQ Penny flip over game . . . 2
1.3 The quantum prisoner's dilemma . . . 2
1.4 The tragedy of the commons . . . 3
1.5 A better scenario: the public goods game . . . 3
1.6 Quantum games . . . 3

1 Quantum games

1.1 History

The field of game theory was started in the 1940s by von Neumann and Oskar Morgenstein.

Game theory is multiperson decision theory in which decision processes are analyzed assuming each player plays rationally to maximize rewards.

Albert Tucker (1950s) invented the Prisoner's Dilemma. Al and Bob allegedly committed a crime and the police interrogate them individually. We know that someone will be sent to jail. Al and Bob could each cooperate (deny the crime and not blame the other) or defect (admit the crime and blame the other). The police will assign sentences ($r = 3$ is reward, $t = 5$ is temptation, and $s = 0$ is the sucker's payoff):

Years in jail (Al, Bob)	Cooperate	Defect
Cooperate	3,3	0,5
Defect	5,0	1,1

(For some reason, we intentionally give a penalty for defecting.)

The dominant strategy is one which earns a player a larger payoff than any other, *regardless* of what the other player does. Which strategy is dominant depends on whether the payoffs are taken to be positive or negative. In any case, the dominant strategy is not optimal.

As a second example, suppose we have two competing companies: Shor's widgets and Chuang's gadgets. The pricing options are:

			Shor's price	
	Profit	\$1	\$2	\$3
	\$1	0,0	50,-10	40,-20
Chuang's price	\$2	-10,50	20,20	90,10
	\$3	-20,40	10,90	50,50

There is no dominant (pure) strategy. There is a Nash equilibrium at (\$1,\$1), though.

A Nash Equilibrium is a set of strategies where no play had an incentive to change his/her action. This is a self-enforcing agreement.

For a third example:

	L	R
U	8,8	0,6
D	6,0	7,7

Now there are two Nash Equilibria. They may not be unique.

For the fourth, there are three players (payoffs are UD, LR, AB):

A	L	R
U	0, 0, 10	-5, -5, 0
D	-5, -5, 0	1, 1, -5

and

B	L	R
U	-2, -2, 0	-5, -5, 0
D	-5, -5, 0	-1, -1, 5

The pure strategy Nash equilibria are ULA and DRB. DRA, however, has interesting properties: if UD and LR collude, then they can drive a cycle DRA →DRB → ULB →ULA → DRB. So Nash equilibria only capture unilateral optima.

1.2 The PQ Penny flip over game

Picard and Q lifelong enemies and they are tossing a coin. The coin starts out tails. First Q gets to flip (turn over) the coin, the Picard can decide whether to flip it, then Q decides again. Q wins on on heads at the end (+1). (Neither play can look at the coin.)

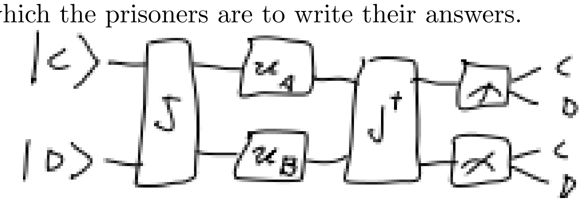
P,Q	NN	NF	FN	FF
N	-1	1	1	-1
F	1	-1	-1	1

There's no pure strategy Nash equilibrium. The mixed strategy Nash equilibrium is for each player to play uniformly at random (there are others, too).

But if Q is quantum and the coin is a qubit. Q plays H. Picard is classical and plays X or I. Then Q plays H and the result is always heads.

1.3 The quantum prisoner's dilemma

There is still no communication between prisoners. Define the states $|C\rangle$ and $|D\rangle$ as the basis for a Hilbert space of decisions (there is one of these qubits per player). In this model, there are two pieces of paper on which the prisoners are to write their answers.



$$J = \frac{II + iXX}{\sqrt{2}} = \exp i\frac{\pi}{4}XX$$

With operators $\hat{C} = I$ and $\hat{D} = X$, note that $\left[J, \hat{C}\hat{C}\right] = \left[J, \hat{D}\hat{D}\right] = 0$, so this is in some sense equivalent to the classical version.

The payoffs are $\$_A = rP_{cc} + pP_{DD} + tP_{DC} + sP_{CD}$ and $\$_B = rP_{cc} + pP_{DD} + sP_{DC} + tP_{CD}$.

The payoff matrix ends up being:

row,col	$\hat{C}$	$\hat{D}$	Z
$\hat{C}$	3,3	0,5	1,1
$\hat{D}$	5,0	1,1	0,5
Z	1,1	5,0	3,3

(The calculations look like, for $\hat{Q}\otimes\hat{C}$, $|\psi\rangle = J^\dagger (ZI) J = (II - iXX)(ZI)(II + iXX)|CC\rangle = YX|CC\rangle = -|DD\rangle$, etc.) $\hat{Q}\hat{Q}$ is now Pareto optimal and a Nash equilibrium.

Issues: what if the prisoners can do any rotation? What if the police use a different set of J operators? And why should the police help the prisoners?

There is a notion that one could replace the prisoners' trust in the police to just trusing quantum mechanics (and possibly some box).

1.4 The tragedy of the commons

This is a proportional game model. Suppose that some proportion of commuters use cars and some use busses. If everyone uses a car, we imagine that it's slow for everyone and, if almost everyone uses busses, the busses go really fast but the cars go faster.

The dominant strategy is for everyone to drive, but that is a bad outcome.

More realistically, suppose that the busses become faster than the cars at some point ($\geq p$ percent of the commuters use cars), due to carpool lanes or whatever. Then the crossover point is a Nash equilibrium (mixed in this case).

We'd like for everyone to take the bus, but the tragedy is that that won't happen.

Traditionally, we try to solve this problem with third-party regulators (toll booths, for example).

1.5 A better scenario: the public goods game

Suppose I give everyone $\$y$ and then pass around a "birthday pool" hat and ask for contributions (of $c_k \leq y$) each. The birthday pool earns interest for awhile and then gets redistributed evenly.

We have n, the number of players; y, the initial endowment; $c_k \leq y$ the individual contribution; and $1 < a < n$, the public gain. The individual payoff is

$$\$_k = (y - c_k) + \frac{a}{n} \sum c_k,$$

and the Nash equilibrium is $c_k = 0 \forall k$. The proof is trivial.

One way to improve it would be to make public the sequence c_k to add social pressure to contribute. This only works for small groups.

Suppose we want to contribute to the Support Madonna for Life fund so that she can continue getting music and so that we can all listen to her music for free. But we need an administrator of the fund and we don't know the people who might or might not contribute.

1.6 Quantum games

This is a scenario that has been worked out.

We have a vendor who makes $J = \frac{I^{\otimes n} + iX^{\otimes n}}{\sqrt{2}}$ boxes. (Open question: can these boxes be verified?)

We start in the state $|c\rangle^{\otimes n}$, run through a J box, then each player applies a controlled unitary gate, send it back through a $J^\dagger$, and the results are all measured. The individual payoff is $\$_k = y - \langle c_k \rangle + \frac{a}{n} \sum \langle c_k \rangle = \langle y \rangle = 1$. The classical Nash equilibrium has $\sum \langle c_k \rangle = 0$. In the quantum scenario, $\langle \$ \rangle = \frac{1+a}{2}$. This is better than the classical result.

For pairwise entanglement, $\langle \$ \rangle = a - 2^{-(n-1)} \approx a$ (quant-ph 0301013).