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Quantum Shannon Theory

Shannon 1948 Proved 2 Thms:

1.) A source S can be compressed to $H(S)$ bits / symbol.

2.) A channel $C: X \to Y$ has $\max_{p(X)} I(X,Y)$ capacity / channel use.

Source S emits symbol s_i w/ probability p_i (s_i independent of s_j $i \neq j$) i.e. the source is memoryless

$H(S) = \sum_i -p_i \log p_i$

<u>Pf.</u> w/ probability $1-\epsilon$ this sequence of symbols is <u>typical</u>.

$2^{n(H(S)+\epsilon)}$ typical sequence $\epsilon \to 0$ as $n \to \infty$

A typical sequence is one whose prob. is between $2^{-n(H(S)+\epsilon)}$ and $2^{-n(H(S)-\epsilon)}$

$$p(s_1, \dots, s_n) = \prod_{i=1}^{n} p_{s_i}$$

$$\log p(\ \ ) = \sum \log p_{s_i}$$

$$E \log p(\ \ ) = E \sum_{i=1}^{n} \log p_{s_i}$$
$$= n \sum_{i=1}^{|S|} p_i \log p_{s_i}$$
$$= -n H(S)$$

In the Quantum version, this becomes:

Source S emits state $|s\rangle$ w/ probability $p(s)$, and is memoryless.

$H(S) = H\left(\sum_{i=1}^{|S|} p_i |s_i\rangle\langle s_i|\right)$ (can also use density matrix instead of pure states but weaken the probability of success)

Alice

$|\otimes s\rangle = |s_1\rangle \otimes |s_2\rangle \otimes |s\rangle \xrightarrow[\text{compress}]{} \underset{\text{qubits}}{nH(S)} \to$ Bob recreate to get ρ.

<u>Def.</u> Fidelity between $|\otimes_n s\rangle$ and ρ is $\langle \otimes_n s | \rho | \otimes_n s\rangle^{1/2}$ ← most common. Sometimes $\sqrt{\ }$ is omitted.

<u>Def.</u> $H(\rho) = -\mathrm{Tr}\, \rho \log \rho = -\sum \lambda_i \log \lambda_i$ $\quad \lambda_i$ e'value of ρ.

<u>Thm.</u> S can be compressed to $(H(S)+\epsilon)n$ qubits and recreated w/ fidelity $1-\epsilon$ where $\epsilon \to 0$ as $n \to \infty$

<u>Pf.</u> Use typical subspace.

Let $\rho_S = \sum_{j=1}^{S} p_i |s_i\rangle\langle s_i|$

Let $\lambda_1, \dots, \lambda_d$, $|v_1\rangle, \dots, |v_d\rangle$ denote the e'values, e'vectors

The typical subspace$^{n,\epsilon}$ is the subspace in basis typical of sequences of $|v_i\rangle$ the probability is given by the e'value. $= \lambda(v_{i_k})$

Typical Subspace $T_S^{(n)}$: $\bigotimes |v_{i_k}\rangle$ w/ $\prod_{k=1}^{n} \rho(v_{i_k})$ between $2^{-(H(S)+\epsilon)n}$ and $2^{-(H(S)-\epsilon)n}$

$\Pi_{T_S^{(n)}}$ projection onto typical subspace

$\mathrm{Tr}\, \Pi_{T_S^{(n)}} \rho_S^{\otimes n} \geq 1-\epsilon$

$\dim \Pi_{T_S^{(n)}} \leq 2^{n(H(S)+\epsilon)}$ = total # of typical subspaces

Compression Algorithm: Project onto $T_{S^{(n)}}$
can express using $H(S+\epsilon)n$ qubits

<u>Thm.</u> Any two sources w/ same density matrix have the same probability of giving outcomes to experiments.

<u>Pf.</u> for POVMs: probability of outcome i for input v is $\mathrm{Tr}\, E_i |v\rangle\langle v|$

If $\rho_1 = \sum_{i=1}^{n} |v_i\rangle\langle v_i| p_i = \sum_{i=1}^{n} |w_i\rangle\langle w_i| q_i$, then the probability of outcome i is

$\mathrm{Tr}\, \rho E_i = \sum p_j \mathrm{Tr}\, |v_j\rangle\langle v_j| E_i = \sum q_j \mathrm{Tr}\, |w_j\rangle\langle w_j| E_i$

Probability of failure of source S is probability of failure of $\tilde{S}$

Send $|v_i\rangle$ w/ prob λ_i.

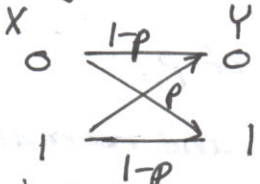
fidelity^2 = $\langle \otimes^n s | \Pi_{T_S^{(n)}} | \otimes^n s \rangle \geq 1-\epsilon$ ← as shown earlier

w/ another ϵ for failure prob. $(E[\langle \otimes^n s | \Pi_{T_S^{(n)}} | \otimes^n s \rangle] \geq 1-\epsilon)$.

Bob gets $\dfrac{\Pi_{T_S^{(n)}} |\otimes^n s\rangle}{1-\epsilon}$ w/ probability $\geq (1-\epsilon)$ $|\text{failure}\rangle$ w/ prob. $\leq \epsilon$

So fidelity is $\langle \otimes^n s |$ Bob's construction$\rangle \geq 1-\epsilon = 1 - O(\epsilon)$
Alice's state
w/ prob $1-2\epsilon$

Suppose you have a classical channel



Stochastic matrix

mutual info. between X and Y is given by

$I(X,Y) = H(Y) - H(Y|X). = H(X) + H(Y) - H(X,Y)$

conditional entropy

joint entropy

$$\text{capacity} = \max_{p(x)} I(Y,X)$$

$$H(Y|X) = \text{Prob}(x=0)\, H\left\{ \begin{matrix} 1-p \\ p \end{matrix} \right\} + \text{Prob}(X=1)\, H\left\{ \begin{matrix} p \\ 1-p \end{matrix} \right\}$$

$$= \text{Prob}(X=0)\,(-p\log p - (1-p)\log(1-p)) + P(X=1)(-p\log p - (1-p)\log(1-p))$$

$H(Y) = H(p\,p(x=1) + (1-p)p(x=0))$ maximized when $p_{(x=0)} = \frac{1}{2}$

$$H(\tfrac{1}{2}) = 1$$

$$H(q) = -q\log q - (1-q)\log(1-q)$$

Accessible Information

max $p(X)$

measurements of outputs

$I(X_i; E_i)$ maximizing over both POVMs and the probability distributions.

transmit one of the 3 states

inner product between any 2 of these states is $-\frac{1}{2}$.

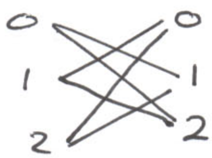
$|0\rangle, -\frac{1}{2}|0\rangle \pm \frac{\sqrt{3}}{2}|1\rangle$

First try
$$\begin{array}{l|l} \frac{1}{3} & |0\rangle \\ \frac{1}{3} & \frac{-|0\rangle}{2} \pm \frac{\sqrt{3}}{2}|1\rangle \end{array}$$

What is the best measurement?

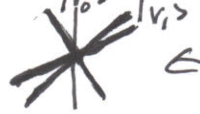
$$E_0 = \frac{2|1\rangle\langle 1|}{3} \qquad E_{1,2} = \frac{2}{3}\left(\frac{-|0\rangle}{2} \pm \frac{\sqrt{3}}{2}|1\rangle\right)\left(-\frac{\langle 0|}{2} \pm \frac{\sqrt{3}}{2}\langle 1|\right)$$

classical channel



$H(Y) - H(Y|X) = \log 3 - \log 2 \approx .58$

Can do better using $|0\rangle$, $-\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$ w/ equal prob $= \frac{1}{2}$



$\leftarrow$ can^{N} measure

Use blocks of length 2

send $|\updownarrow\updownarrow\rangle$, $|\swarrow\!\!\nearrow\swarrow\!\!\nearrow\rangle$, $|\nwarrow\!\!\searrow\nwarrow\!\!\searrow\rangle$ get 1.369 for 2 measurements.

$\frac{1}{3}$ $\frac{1}{3}$ $\frac{1}{3}$

as opposed to .645 access info

3 dim subspace of 4 dim (2 qubits)

Thm. If you send w/ states $\rho_1, \rho_2, \dots, \rho_n$

marginal probs $p_1, \dots, p_n$, kth codeword

$\exists$ codewords $\left\{ \rho_{i_1}^k \otimes \rho_{i_2}^k \otimes \cdots \otimes \rho_{i_n}^k = |\psi_k\rangle \right\}$ $2^{I(X,Y)n}$

Joint measurement

so $I(X,Y) \geq n\left[H\left(\sum p_i \rho_i\right) - \sum p_i H(\rho_i) - \epsilon \right]$ where $\epsilon \to 0$ as $n \to \infty$

$\|$

χ Holevo information (or Holevo capacity)

Capacity of $\updownarrow$

$$\frac{1}{3}(|0\rangle\langle 0|) + \frac{1}{3}\left(-|0\rangle + \frac{\sqrt{3}}{2}|1\rangle\right)\left(-\frac{\langle 0|}{2} - \quad\right) + \frac{1}{3}(\quad)(\quad) = \begin{pmatrix} \frac{1}{2} & \\ & \frac{1}{2} \end{pmatrix}$$

$$H\begin{pmatrix} 1/2 & \\ & 1/2 \end{pmatrix} = 1$$