WHAT IS AN L-FUNCTION?

This handout is extracted from a forthcoming paper by David Farmer, Ameya Pitale, Nathan Ryan, and Ralf Schmidt.

0.1. **Tempered balanced analytic L-functions.**

Throughout the axioms, $s = \sigma + it$ is a complex variable with σ and t real.

A ***tempered balanced analytic L-function*** is a function $L(s)$ which satisfies the three axioms below.

Axiom 1 (Analytic properties): $L(s)$ is given by a Dirichlet series

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \tag{0.1}$$

where $a_n \in \mathbb{C}$.

a) *Convergence:* $L(s)$ converges absolutely for $\sigma > 1$.

b) *Analytic continuation:* $L(s)$ continues to a meromorphic function having only finitely many poles, with all poles [[in $\sigma > 0$]] lying on the $\sigma = 1$ line.

Axiom 2 (Functional equation): There is a positive integer N called the ***conductor*** of the L-function, a positive integer d called the ***degree*** of the L-function, a pair of non-negative integers (d_1, d_2) called the ***signature*** of the L-function, where $d = d_1 + 2d_2$, and complex numbers $\{\mu_j\}$ and $\{\nu_j\}$ called the ***spectral parameters*** of the L-function, such that the ***completed*** L***-function***

$$\Lambda(s) = N^{s/2} \prod_{j=1}^{d_1} \Gamma_{\mathbb{R}}(s + \mu_j) \prod_{j=1}^{d_2} \Gamma_{\mathbb{C}}(s + \nu_j) \cdot L(s) \tag{0.2}$$

has the following properties:

a) *Bounded in vertical strips:* Away from the poles of the L-function, $\Lambda(s)$ is bounded in vertical strips $\sigma_1 \leq \sigma \leq \sigma_2$.

b) *Functional equation:* There exists $\varepsilon \in \mathbb{C}$, called the ***sign*** of the functional equation, such that

$$\Lambda(s) = \varepsilon \overline{\Lambda}(1 - s). \tag{0.3}$$

c) *Selberg bound:* For every j we have $\mathrm{Re}(\mu_j) \in \{0, 1\}$ and $\mathrm{Re}(\nu_j) \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, ...\}$.

d) *Balanced:* We have $\mathrm{Im}\left(\sum \mu_j + \sum (2\nu_j + 1)\right) = 0$.

Axiom 3 (Euler product): There is a product formula

$$L(s) = \prod_{p\,\mathrm{prime}} F_p(p^{-s})^{-1}, \tag{0.4}$$

absolutely convergent for $\sigma > 1$.

a) *Polynomial:* For every prime p, the F_p is a polynomial of degree at most d, with $F_p(0) = 1$.

b) *Central character:* There exists a Dirichlet character χ mod N, called the ***central character*** of the L-function, such that

$$F_p(z) = 1 - a_p z + \cdots + (-1)^d \chi(p) z^d. \tag{0.5}$$

c) *Parity:* The spectral parameters determine the parity of the central character:

$$\chi(-1) = (-1)^{\sum \mu_j + \sum (2\nu_j + 1)}. \tag{0.6}$$

d) *Ramanujan bound:* Write F_p in factored form as

$$F_p(z) = (1 - \alpha_{1,p} z) \cdots (1 - \alpha_{d_p,p} z) \tag{0.7}$$

with $\alpha_{j,p} \neq 0$. If $p \nmid N$ then $|\alpha_{j,p}| = 1$ for all j. If $p \mid N$ then $|\alpha_{j,p}| = p^{-m_j/2}$ for some $m_j \in \{0, 1, 2, ...\}$, and $\sum m_j \leq d - d_p$.

0.1.1. *Comments on the terminology.* The term ***balanced*** is described by Axiom 2d). The summand "+1" can obviously be omitted from the condition, but we include it for uniformity with Axiom 3c). If we omit the modifier 'balanced' when describing an L-function, then we mean a function of the form $L(s + iy)$ where $L(s)$ is balanced and $y \in \mathbb{R}$. If $L(s)$ is a (not necessarily balanced) L-function, then it is straightforward to check that there exists exactly one $y_0 \in \mathbb{R}$ such that $L(s + iy_0)$ is balanced.

The term ***tempered*** refers to both the Selberg bound (Axiom 2c) and the Ramanujan bound (Axiom 3d). Neither bound has been proven for most automorphic L-functions, but if those axioms fail for an automorphic L-function, they must fail in a specific way arising from the fact that the underlying representation in unitary. A precise description of the possibilities is given by the ***unitary pairing condition***.

In the functional equation, the function $\overline{\Lambda}$ is the Schwartz reflection of Λ, defined for arbitrary analytic functions f by $\overline{f}(z) = \overline{f(\overline{z})}$. The tuple $(\varepsilon, N, \{\mu_1, \ldots, \mu_J\}, \{\nu_1, \ldots, \nu_K\})$ is the ***functional equation data*** of the L-function.

In the Euler product, the polynomials F_p are known as the ***local factors***, and the reciprocal roots $\alpha_{j,p}$ are called the ***Satake parameters*** at p. If $p \mid N$ then we say p is a ***bad*** prime, and if $p \nmid N$ then p is ***good***. By Axiom 3b), if p is good then d_p, the degree of the local factor at p, equals d, and if p is bad then $d_p < d$.

It follows straight from the definition that if $L_1(s)$ and $L_2(s)$ are analytic L-functions then so is $L_1(s)L_2(s)$. And if both L_1 and L_2 are balanced, or tempered, then so is their product. If the analytic L-function $L(s)$ cannot be written nontrivially as $L(s) = L_1(s)L_2(s)$, then we say that L is ***primitive***. Here 'nontrivially' refers to the fact that the constant function 1 is a degree 0 L-function. It follows from the Selberg orthogonality that L-functions factor uniquely into primitive L-functions:

Theorem 0.8 (Conrey and Ghosh)**.** *Assume the Selberg orthogonality conjecture. If $L(s)$ is an analytic L-function then*

$$L(s) = L_1(s) \cdots L_n(s) \tag{0.9}$$

where each L_j is a nontrivial primitive analytic L-function. The representation is unique except for the order of the factors.

Selberg conjectured that two primitive L-functions which are not equal must be "orthonormal" in the following sense.

Conjecture 0.10 (Selberg). *If $L_1(s) = \sum a_n n^{-s}$ and $L_2(s) = \sum b_n n^{-s}$ are primitive analytic L-functions, then*

$$\sum_{p \le X} \frac{a_p \overline{b_p}}{p} = (\delta_{L_1,L_2} + o(1)) \log\log X, \tag{0.11}$$

where δ is the Kronecker delta:

$$\delta_{a,b} = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise.} \end{cases} \tag{0.12}$$

The conjecture has important consequences. For example, if $L(s) = L_1^{n_1}(s) \cdots L_m^{n_m}(s) = \sum a_n n^{-s}$ with $L_j(s)$ primitive, then

$$\sum_{p \le X} \frac{|a_p|^2}{p} \sim (n_1^2 + \cdots + n_m^2) \log\log X. \tag{0.13}$$

So by looking at the average value of $|a_p|^2$ for a given L-function, you can tell if it is primitive, and if it isn't, then you know something about how it might factor.

If two L-functions have enough local factors in common, then they must actually be the same function. The following is a weak version of this result, but it is sufficient for this discussion.

Theorem 0.14 (Strong Multiplicitly One (SMO)). *If*

$$L_1(p) = \prod_p F_{1,p}(p^{-s})^{-1} \qquad \text{and} \qquad L_2(p) = \prod_p F_{2,p}(p^{-s})^{-1} \tag{0.15}$$

are L-functions, and $F_{1,p}(z) = F_{2,p}(z)$ for all but finitely many primes p, then $L_1(s) = L_2(s)$. In particular, the bad local factors, conductor, sign, and Γ-factors are all determined by the good local factors.

0.2. **Operations on L-functions.** We describe ways to turn an L-function, or a collection of L-functions, into another L-function. For automorphic L-functions, many of these operations have been shown to produce L-functions. We describe the operations purely in terms of the axioms for analytic L-functions, but in many cases the analytic continuation and functional equation of the resulting object are conjectural.

In this section we generally omit the adjectives 'tempered,' 'balanced,' 'analytic,' and 'arithmetic,' with the understanding that whatever properties apply to the input L-function(s) also apply to the output L-functions(s).

0.2.1. *Dual.* If $L(s) = \sum a_n n^{-s}$ is an L-function then so is the ***dual*** L-function $\overline{L}(s) = \sum \overline{a_n} n^{-s}$. If L has functional equation data $(\varepsilon, N, \{\mu_1, \ldots, \mu_J\}, \{\nu_1, \ldots, \nu_k\})$ then $\overline{L}$ has functional equation data $(\overline{\varepsilon}, N, \{\overline{\mu_1}, \ldots, \overline{\mu_J}\}, \{\overline{\nu_1}, \ldots, \overline{\nu_k}\})$ An L-function $L(s)$ is ***self-dual*** if $L(s) = \overline{L}(s)$, which is equivalent to the L-function having real coefficients.

0.2.2. *Rankin-Selberg convolution.* Suppose

$$L_1(s) = \sum a_n n^{-s} = \prod_p \frac{1}{(1 - \alpha_{1,p} p^{-s}) \cdots (1 - \alpha_{d_1,p} p^{-s})} \tag{0.16}$$

and

$$L_2(s) = \sum b_n n^{-s} = \prod_p \frac{1}{(1-\beta_{1,p}p^{-s})\cdots(1-\beta_{d_2,p}p^{-s})} \tag{0.17}$$

are L-functions of degrees d_1 and d_2, respectively. Recall that the above local factors are valid for all p, but if p is a bad prime then some of the $\alpha_{j,p}$, respectively $\beta_{j,p}$, are zero.

The Rankin-Selberg convolution of L_1 and L_2, denoted $L_1 \times L_2$, is given by an Euler product whose good local factors are given by

$$F_{L_1\times L_2,p}(z) = \prod_{j=1}^{d_1}\prod_{k=1}^{d_2} \frac{1}{1-\alpha_{j,p}\beta_{k,p}z}. \tag{0.18}$$

If p is good for both L_1 and L_2, then it is good for $L_1 \times L_2$. If $L_1 \times L_2$ is an L-function, then by SMO the bad Euler factors and the functional equation parameters are uniquely determined. There is no simple recipe to determine all those invariants of $L_1 \times L_2$ directly from the data for L_1 and L_2, but the determination of the Γ-factors is relatively straight-forward, as is the determination of those bad factors which have a particularly simple form.

0.2.3. *Symmetric and exterior powers.* If

$$L(s) = \sum a_n n^{-s} = \prod_p \frac{1}{(1-\alpha_{1,p}p^{-s})\cdots(1-\alpha_{d,p}p^{-s})} \tag{0.19}$$

is an L-function of degree d, then (conjecturally) we can make new L-functions from it:

The ***exterior square L-function***:

$$L^S(s,\mathrm{ext}^2) = \prod_{p \text{ good}} \prod_{1\le j<k\le d} (1-\alpha_j\alpha_k p^s)^{-1}, \tag{0.20}$$

which has degree $d(d-1)/2$, and the ***symmetric square L-function***:

$$L^S(s,\mathrm{sym}^2) = \prod_{p \text{ good}} \prod_{1\le j\le k\le d} (1-\alpha_j\alpha_k p^s)^{-1}, \tag{0.21}$$

which has degree $d(d+1)/2$. Here the partial L-function $L^S(s)$ is $L(s)$ with the bad factors removed. By SMO, $L^S(s)$ determines (in a non-effective way) everything about the L-function.

Note that $L \times L(s) = L(s,\mathrm{ext}^2)L(s,\mathrm{sym}^2)$.

It might look like $L(s,\mathrm{ext}^2)$ is a factor of $L(s,\mathrm{sym}^2)$, since the good local factors of $L(s,\mathrm{ext}^2)$ appear within the good local factors of $L(s,\mathrm{sym}^2)$. But things don't work that way, and usually $L(s,\mathrm{sym}^2)/L(s,\mathrm{ext}^2)$ will have a pole at each zero of $L(s,\mathrm{ext}^2)$. Most Euler products are not L-functions.

One can also take higher symmetric and exterior powers.

0.3. **Exercises.** If p is good then
$a_p(L_1 \times L_2) = a_p(L_1)a_p(L_2)$
$a_p(\mathrm{ext}^2) = a_p^2 - a_{p^2} =$ the coefficient of z^2 in $F_p(z)$
$a_p(\mathrm{sym}^2) = a_{p^2}$